Applications of the GPAT to triangle geometry

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Abstract

Through use of the generalized parallel axis theorem (GPAT) and related algebraic expressions, results on the variance of weighted collections of points can be applied to the recreation of geometric properties of the triangle, with minimal appeal to geometric arguments.

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1 Preliminaries

1.1 Barycentric Coordinates

Much of the following computation is completed in the context of barycentric coordinates. A reference triangle of vertices A, B, C and side lengths a, b, c assigned in the usual fashion is fixed in the plane. A point P in the plane is then uniquely determined (up to multiplication by a scalar) as a triple of points (P_1, P_2, P_3) such that P is the centre of mass of the triangle ABC when weights P_1, P_2 and P_3 are assigned to A,B and C respectively. To simplify calculations and for ease of comparison, it is often helpful to work with normalized coordinates, such that $P_1 + P_2 + P_3 = 1$, giving a unique representation for P. In any case, we require $P_1 + P_2 + P_3 \neq 0$, which of course holds if the coordinates have been normalized.

1.2 Useful formulae and methods

For a mathematical introduction to this section, see [1]. A number of calculations were automated through use of the Computer Algebra System *Maple*. Throughout, a reference triangle of sides a, b, c is assumed. Firstly, the mean square distance, variance and the determination of the squared distance between two arbitrary points through use of the GPAT:

```
> dsq:=proc(P1,P2,P3,Q1,Q2,Q3)
  (P1*Q2*c^2 + P1*Q3*b^2 + P2*Q1*c^2 + P2*Q3*a^2 + P3*Q1*b^2 + P3*Q2*a^2)
  /((P1+P2+P3)*(Q1+Q2+Q3));
  end:

> varsig:=proc(P1,P2,P3)
  dsq(P1,P2,P3,P1,P2,P3)/2;
  end:

> GPATdistsq:=proc(P1,P2,P3,Q1,Q2,Q3)
  dsq(P1,P2,P3,Q1,Q2,Q3)-varsig(P1,P2,P3)-varsig(Q1,Q2,Q3)
  end:
```

The procedure `dsq`, given two points $P = (P_1, P_2, P_3)$ and $Q = (Q_1, Q_2, Q_3)$, returns the mean square distance of the triangles Δ_P, Δ_Q, where Δ_P is the reference triangle ABC with weights P_1, P_2, P_3 at the vertices. This is calculated as

$$d^2(\Delta_P, \Delta_Q) = \frac{P_1Q_1c^2 + P_1Q_3b^2 + P_2Q_1c^2 + P_2Q_3a^2 + P_3Q_1b^2 + P_3Q_2a^2}{(P_1 + P_2 + P_3)(Q_1 + Q_2 + Q_3)}$$

The procedure `varsig`, given a point (P_1, P_2, P_3), computes

$$\sigma^2(\Delta_P) = d^2(\Delta_P, \Delta_P)/2$$

the variance of Δ_P (this identity is a corollary of the GPAT). Finally, we can use the GPAT to recover $(PQ)^2$, since P and Q are the centres of mass of Δ_P, Δ_Q respectively and hence

$$d^2(\Delta_P, \Delta_Q) = \sigma^2(\Delta_P) + (PQ)^2 + \sigma^2(\Delta_Q)$$

So `GPATdistsq` calculates

$$(PQ)^2 = d^2(\Delta_P, \Delta_Q) - \sigma^2(\Delta_P) - \sigma^2(\Delta_Q)$$

Given three points not all colinear, it is possible to find a circle upon which they all lie. This task is equivalent to finding a point equidistant from all three, which will then be the centre of the circle. Given three points $P = (P_1, P_2, P3)$, $Q = (Q_1, Q_2, Q_3)$ and $R = (R_1, R_2, R_3)$, a hypothesised common centre $O = (x, y, z)$ will satisfy

$$(OP)^2 - (OQ)^2 = (OP)^2 - (OR)^2 = (OQ)^2 - (OR)^2 = 0$$

That is, three equations which suffice to determine the three unknowns x, y, z (provided P,Q,R are not colinear). Such a calculation is entirely routine with *Maple*, and so we offer two procedures to determine the centre- the second one imposes the additional restraint $x+y+z=1$, that is, returns a normalized description of O:

```
>commoncentre:=proc(P1,P2,P3,Q1,Q2,Q3,R1,R2,R3)
 solve({GPATdistsq(P1,P2,P3,x,y,z)-GPATdistsq(Q1,Q2,Q3,x,y,z)=0,
        GPATdistsq(P1,P2,P3,x,y,z)-GPATdistsq(R1,R2,R3,x,y,z)=0,
        GPATdistsq(Q1,Q2,Q3,x,y,z)-GPATdistsq(R1,R2,R3,x,y,z)=0},{x,y,z});
 end:
```

```
>scaledcommoncentre:=proc(P1,P2,P3,Q1,Q2,Q3,R1,R2,R3)
 solve({GPATdistsq(P1,P2,P3,x,y,z)-GPATdistsq(Q1,Q2,Q3,x,y,z)=0,
        GPATdistsq(P1,P2,P3,x,y,z)-GPATdistsq(R1,R2,R3,x,y,z)=0,
        GPATdistsq(Q1,Q2,Q3,x,y,z)-GPATdistsq(R1,R2,R3,x,y,z)=0,
        x+y+z=1},{x,y,z});
 end:
```

Supplying this function with entirely arbitrary points returns an uninspiring response spanning several pages, but typical input causes the vast majority of terms to cancel, as shall be demonstrated.

Further, knowing the centre and any point on the circle, we can find the square of the radius. Whilst this is simply `GPATdistsq` again, it is helpful to define a procedure with a more intuitive name for use in further calculation. Hence we define

```
>commonradiussq:=proc(P1,P2,P3,x,y,z)
 dsq(P1,P2,P3,x,y,z)-varsig(P1,P2,P3)-varsig(x,y,z):
 simplify(%);
 end:
```

2 Applications: Circumcentres and Circumradii

2.1 The circumcircle

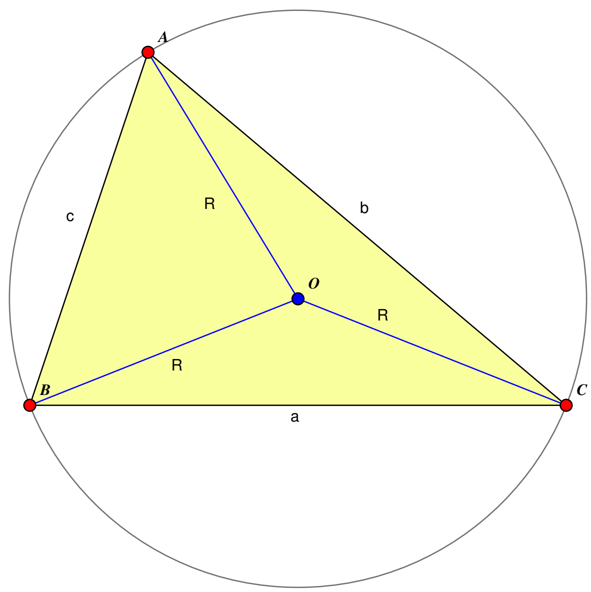

Figure 1: The Circumcentre [8]

A natural first choice of points to work with are the vertices A, B, C of the reference triangle itself. Thus we will find the circumcircle, determining its centre (the circumcentre O) in barycentric coordinates (a result not easily found by direct appeal to geometry) and verifying the circumradius R. Normalizing, our points are $A = (1,0,0),\ B = (0,1,0),\ C = (0,0,1)$. Plugging directly into `commoncentre` we obtain

$$x = \frac{-a^2(c^2+b^2-a^2)z}{c^2(-b^2+c^2-a^2)},\ y = \frac{-zb^2(a^2-b^2+c^2)}{c^2(-b^2+c^2-a^2)},\ z = z$$

Which (since these are homogeneous coordinates) suggests a more natural rendition as

$$(a^2(b^2+c^2-a^2), b^2(c^2+a^2-b^2), c^2(a^2+b^2-c^2))$$

Note that the procedures are robust to unnormalized inputs- taking A as (2,0,0) or C as (0,0,c) does not alter the result. Normalizing this expression for the circumcentre, we obtain

$$O = \frac{1}{\sum_{cyc}(2a^2b^2-a^4)}(a^2(b^2+c^2-a^2), b^2(c^2+a^2-b^2), c^2(a^2+b^2-c^2)) \tag{1}$$

This denominator routinely arises in such calculations, since it is related, by Heron's formula, to the area of the triangle:

$$\begin{aligned}
\sum_{cyc}(2a^2b^2 - a^4) &= 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 \ , \ \textit{expanding} \\
&= (a+b+c)(-a+b+c)(a-b+c)(a+b-c) \ , \ \textit{factoring} \\
&= 16\frac{(a+b+c)}{2}\frac{(-a+b+c)}{2}\frac{(a-b+c)}{2}\frac{(a+b-c)}{2} \\
&= 16s(s-a)(s-b)(s-c) \ , \ \textit{for } s = (a+b+c)/2 \textit{ the semiperimeter} \\
&= 16\Delta^2 \ , \ \textit{Where } \Delta \textit{ denotes the area of triangle } ABC
\end{aligned}$$

The useful equivalencies are

$$\sum_{cyc}(2a^2b^2 - a^4) = (a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 16\Delta^2 \tag{2}$$

In particular, it arises when we consider the circumradius R. Using `commonradiussq` with arguments $(1,0,0)$ and (x,y,z) with the latter extracted from `scaledcommoncentre` we deduce

$$R^2 = \frac{-c^2a^2b^2}{-2a^2c^2 - 2a^2b^2 + a^4 + b^4 - 2c^2b^2 + c^4}$$

Or more cleanly

$$R^2 = \frac{a^2b^2c^2}{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4} = \frac{a^2b^2c^2}{16\Delta^2}$$

So we have recovered the geometric property that

$$R = \frac{abc}{4\Delta}$$

2.2 The Nine-point circle

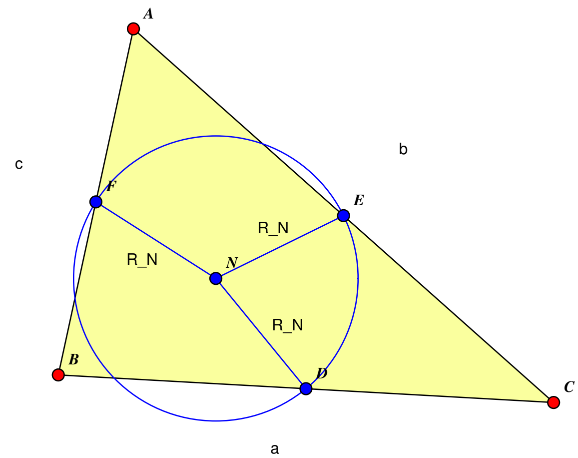

Figure 2: The Nine Point Centre

We apply the *Maple* procedures to points $D = (0, \frac{1}{2}, \frac{1}{2})$, $E = (\frac{1}{2}, 0, \frac{1}{2})$, $F = (\frac{1}{2}, \frac{1}{2}, 0)$; namely the (normalized) midpoints of the triangle sides. This generates the barycentric coordinates for the nine point centre which, if normalized, are as follows:

$$N = \frac{1}{2\sum_{cyc}(2a^2b^2 - a^4)}(2b^2c^2+a^2c^2+a^2b^2-b^4-c^4, 2a^2c^2+b^2c^2+a^2b^2-a^4-c^4, 2a^2b^2+a^2c^2+b^2c^2-a^4-b^4) \tag{3}$$

Using `commonradiussq` we determine R_N, the radius of the nine point circle, (equivalently, the circumradius of the medial triangle of reference triangle ABC) to satisfy

$$(R_N)^2 = \frac{a^2b^2c^2}{4\sum_{cyc}(a^2b^2 - a^4)}$$

i.e.

$$R_N = R/2$$

2.3 The Anticomplementary circle

In the previous section we considered the medial triangle of ABC; now we direct our attention to the triangle for which ABC is the medial triangle. This is the Anticomplimentary triangle, having (normalized) vertices at $(-1,1,1)$, $(1,-1,1)$ and $(1,1,-1)$. We immediately expect (and obtain) a circumradius twice that of ABC; the centre of this circle is more interesting. This turns out to be

$$\frac{1}{\sum_{cyc}(2a^2b^2-a^4)}((a^2+c^2-b^2)(a^2+b^2-c^2),(b^2+c^2-a^2)(b^2+a^2-c^2),(a^2+c^2-b^2)(b^2+c^2-a^2) \quad (4)$$

Geometrically, the centre of the Anticomplimentary circle is the orthocentre H of the reference triangle ABC (see figure 3). Thus the above expression gives a description in barycentric coordinates of H.

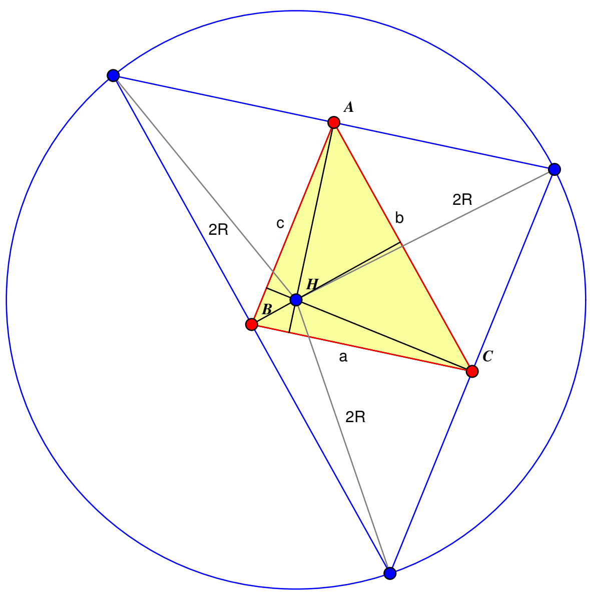

Figure 3: The centre of the Anticomplementary circle is the orthocentre of ΔABC

2.4 The Incentral Circle

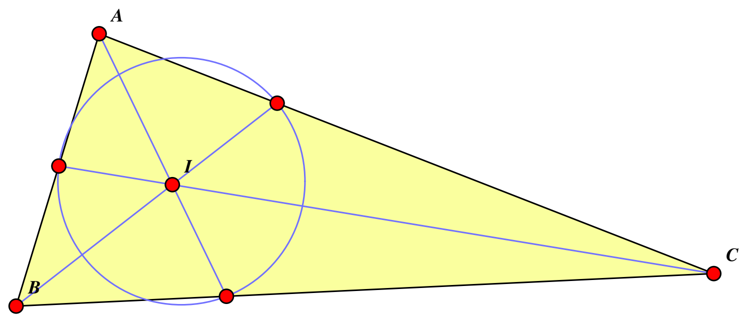

Figure 4: The Incentral circle (cevians through I)

We include this as a curiosity and to demonstrate a more intricate result that can be obtained purely through algebraic means with the GPAT. The Incentral Triangle is the cevian triangle of ABC with respect to the Incenter I, with vertices $(0, b, c)$, $(a, 0, c)$ and $(a, b, 0)$. We generate the common centre, which is uninteresting, and proceed to find the square of the radius which, when factorised, is

$$\frac{\left(b^3-cb^2-c^2b+c^3+ab^2-3\,bac+ac^2-ba^2-ca^2-a^3\right)\left(-b^3-cb^2+c^2b+c^3+ab^2+3\,bac+ac^2+ba^2-ca^2-a^3\right)\left(-b^3-cb^2+c^2b+c^3-ab^2-3\,bac-ac^2+ba^2-ca^2+a^3\right)bac}{-4(b+c)^2(a+b)^2(a+c)^2(b+c+a)(-b+c-a)(b+c-a)(a-b+c)}$$

Defining $f(a, b, c) = a^3 - ba^2 + ca^2 - b^2a - c^2a - 3bca + b^3 - c^3 - bc^2 + b^2c$, this simplifies considerably to

$$\frac{f(a,b,c)f(b,c,a)f(c,a,b)abc}{4(b+c)^2(a+b)^2(a+c)^2(a+b+c)(b+c-a)(a+c-b)(a+b-c)}$$

Which we recognise as

$$\frac{f(a,b,c)f(b,c,a)f(c,a,b)abc}{4(b+c)^2(a+b)^2(a+c)^2 16\Delta^2}$$

Hence the radius is elegantly captured by the expression

$$\frac{\sqrt{abcf(a,b,c)f(b,c,a)f(c,a,b)}}{8\Delta(a+b)(a+c)(b+c)}$$

3 Further Applications

3.1 Determination of the Contact triangle

When the radius and centre of a circle are easily determined geometrically, it may be profitable to use this information to reconstruct the circle or key points upon it. One such example is the contact triangle- this has vertices at the intersection of the incircle with the reference triangle (hence, the incircle is its circumcircle). A very easy geometric argument gives the incentral radius r via

$$r^2 = \frac{(b+c-a)(c+a-b)(a+b-c)}{4(a+b+c)}$$

whilst it is not too difficult to determine the barycentric coordinates of the incenter as being $I = (a, b, c)$. (See, for instance, [4]) Thus the vertices P, Q, R of the contact triangle will satisfy $(IP)^2 = (IQ)^2 = (IR)^2 = r^2$ and, since they lie on the sides of the reference triangle, one coordinate for each will be zero.

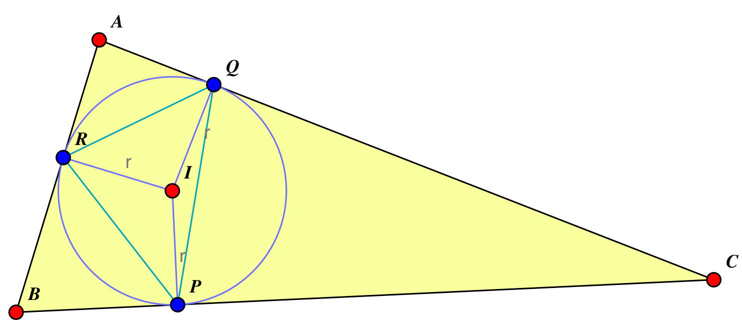

Figure 5: The Contact Triangle

With access to the GPAT procedures, this turns out to be sufficient to generate the barycentric coordinates of the points. We consider $P = (P_1, P_2, P_3)$ and compute LHS which is `GPATdistsq`$(P1, P2, P3, a, b, c)$,

$$\frac{P_1bc^2 + P_1cb^2 + P_2ac^2 + P_2ca^2 + P_3ab^2 + P_3ba^2}{(P_1+P_2+P_3)(a+b+c)} - \frac{2P_1P_2c^2 + 2P_1P_3b^2 + 2P_2P_3a^2}{2(P_1+P_2+P_3)^2} - \frac{2abc^2 + 2acb^2 + 2bca^2}{2(a+b+c)^2}$$

and RHS which is r^2 or

$$\frac{(b+c-a)(c+a-b)(a+b-c)}{4(a+b+c)}$$

Setting P_1 to be 0 (that is, seeking the vertex on BC), we can simplify $LHS - RHS$ to

$$\frac{(P_2c + P_3c + aP_2 - aP_3 - P_2b - P_3b)^2}{4(P_2 + P_3)^2}$$

so to generate a normalized description we solve for this equal to 0 and $P_2 + P_3 = 1$, yielding

$$P = (0, \frac{a+b-c}{2a}, \frac{a-b+c}{2a})$$

By an entirely similar process, we recover

$$Q = (\frac{a+b-c}{2b}, 0, \frac{-a+b+c}{2b})$$

and

$$R = (\frac{a-b+c}{2c}, \frac{-a+b+c}{2c}, 0)$$

3.2 The Euler Line

In [3], the equation of a line joining two points P,Q with barycentric coordinates (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) respectively is given as

$$\begin{vmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0$$

Hence, by properties of the determinant, any point $R = \lambda P + \mu Q$, that is with Barycentric coordinates $(\lambda P_1 + \mu Q_1, \lambda P_2 + \mu Q_2, \lambda P_3 + \mu Q_3)$ will be colinear with P and Q. Using the *Maple* procedures developed earlier, we calculate

$$d_1 = \texttt{GPATdistsq}(P_1, P_2, P_3, \lambda P_1 + \mu Q_1, \lambda P_2 + \mu Q_2, \lambda P_3 + \mu Q_3)$$

$$d_2 = \texttt{GPATdistsq}(Q_1, Q_2, Q_3, \lambda P_1 + \mu Q_1, \lambda P_2 + \mu Q_2, \lambda P_3 + \mu Q_3)$$

Then

$$\frac{d_1}{d_2} = \frac{\mu^2(Q_1 + Q_2 + Q_3)^2}{\lambda^2(P_1 + P_2 + P_3)^2}$$

Hence, if P and Q have normalised descriptions in barycentric coordinates as (P_1, P_2, P_3) and (Q_1, Q_2, Q_3) and $R = (\lambda P_1 + \mu Q_1, \lambda P_2 + \mu Q_2, \lambda P_3 + \mu Q_3)$ for positive λ, μ then

$$\frac{(PR)^2}{(RQ)^2} = \frac{\mu^2}{\lambda^2}$$

and so

$$PR : RQ = \mu : \lambda$$

From the earlier circumcentre calculations we may now confirm results about the Euler Line. From (1) the Circumcentre O has barycentric coordinates

$$\frac{1}{\sum_{cyc}(2a^2b^2 - a^4)}(a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2))$$

and (4) gives the Orthocentre H as

$$\frac{1}{\sum_{cyc}(2a^2b^2 - a^4)}((a^2 + c^2 - b^2)(a^2 + b^2 - c^2), (b^2 + c^2 - a^2)(b^2 + a^2 - c^2), (a^2 + c^2 - b^2)(b^2 + c^2 - a^2)$$

We can therefore construct their midpoint J as $\frac{1}{2}O + \frac{1}{2}H$. Considering the first component J_1, we obtain

$$\begin{aligned} J_1 &= \frac{1}{\sum_{cyc}(2a^2b^2 - a^4)}(a^2(b^2 + c^2 - a^2)/2 + (a^2 + c^2 - b^2)(a^2 + b^2 - c^2)/2) \\ &= \frac{1}{2\sum_{cyc}(2a^2b^2 - a^4)}(a^2(b^2 + c^2 - a^2) + a^4 - b^4 - c^4 + 2b^2c^2) \\ &= \frac{1}{2\sum_{cyc}(2a^2b^2 - a^4)}(a^2b^2 + a^2c^2 - a^4 + a^4 - b^4 - c^4 + 2b^2c^2) \\ &= \frac{1}{2\sum_{cyc}(2a^2b^2 - a^4)}(2b^2c^2 + a^2c^2 + a^2b^2 - b^4 - c^4) \end{aligned}$$

and, by cyclic permutation of arguments

$$J_2 = \frac{1}{2\sum_{cyc}(2a^2b^2 - a^4)}(2a^2c^2 + b^2c^2 + a^2b^2 - a^4 - c^4)$$

$$J_3 = \frac{1}{2\sum_{cyc}(2a^2b^2 - a^4)}(2a^2b^2 + a^2c^2 + b^2c^2 - a^4 - b^4)$$

We recognise J from (3), the position of the Nine-point centre. That is, we have confirmed that N is a linear combination of O and H, and hence O, N, H are colinear with $ON{=}NH$, i.e., N is the midpoint of OH.

The centroid G is easily determined to have Barycentric coordinates $(1, 1, 1)$ (see, for instance, [4]) and so we may attempt to recover the result that $OG : GH = 1 : 2$; The previous analysis suggests we should recover G as the point $R = O + \frac{1}{2}H$. Considering as before the first component we get

$$\begin{aligned}
R_1 &= \frac{1}{\sum_{cyc}(2a^2b^2 - a^4)}(a^2(b^2 + c^2 - a^2) + (a^2 + c^2 - b^2)(a^2 + b^2 - c^2)/2) \\
&= \frac{1}{\sum_{cyc}(2a^2b^2 - a^4)}(a^2b^2 + a^2c^2 - \frac{a^4}{2} - \frac{b^4}{2} - \frac{c^4}{2} + b^2c^2) \\
&= \frac{1}{\sum_{cyc}(2a^2b^2 - a^4)}(\frac{1}{2}(a + b + c)(b + c - a)(a + c - b)(a + b - c)) \\
&= \frac{1}{\sum_{cyc}(2a^2b^2 - a^4)}(\frac{1}{2}\sum_{cyc}(2a^2b^2 - a^4) \text{ from (2)} \\
&= \frac{1}{2}
\end{aligned}$$

Again, cyclic permutation of the arguments recovers R_2 and R_3, and so

$$R = O + \frac{1}{2}H = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$$

Which, when normalised, gives the Centroid $G = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ as desired. Since this point was obtained by a linear combination of O and H, it too is necessarily colinear with them.

We have obtained the results

$$\begin{gathered}
O, G, N, H \text{ are colinear} \\
ON : NH = 1 : 1 \\
OG : GH = 1 : 2
\end{gathered}$$

4 Constraints on points

In keeping with [2], we turn our attention to constraints on the relative positions of triangle centres within the context of the Brocard disc (the disc on diameter OS where S is the Symmedian point). Experimentation with the *Cinderella* [8] dynamic geometry package gives rise to some conjectures on non-equilateral, non-degenerate triangles.

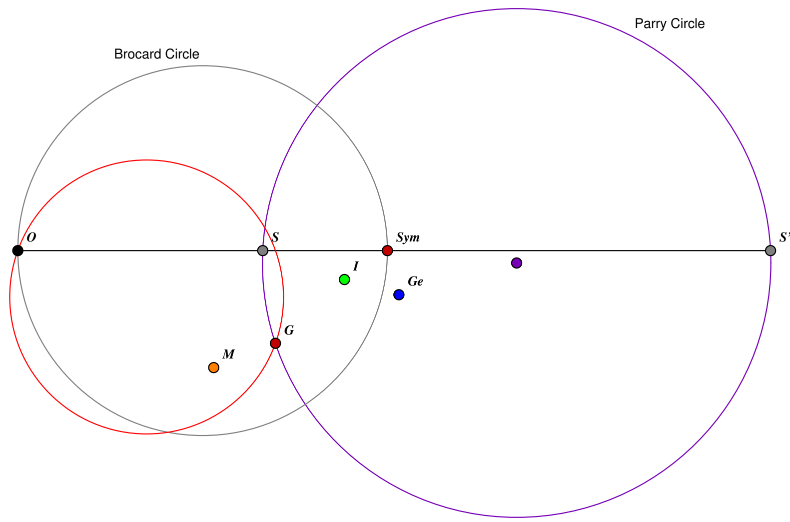

Figure 6: The Brocard circle and points of interest *Circumcentre O, Symmedian point Sym, Isodynamic Points S and S′, Centroid G, Mittenpunkt M, Incenter I, Gergonne Point Ge.*

4.1 The Mittenpunkt

Conjectures:

1. The Mittenpunkt is constrained to the Brocard disc (grey in Fig. 6)
2. The Mittenpunkt is constrained to the disc on diameter OG (red in Fig. 6)
3. The Mittenpunkt cannot lie in the Parry circle (purple in Fig. 6)

Working with the usual *Maple* procedures, a computer proof of conjecture 2 can be established as follows. Consider the midpoint M' of diameter OG. By the methods discussed in section 3.2

this will have coordinates

$$M_1' = \frac{(4a^4 + b^4 + c^4 - 2b^2c^2 - 5a^2b^2 - 5a^2c^2}{6(a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2)}$$

$$M_2' = \frac{4b^4 + a^4 + c^4 - 2a^2c^2 - 5a^2b^2 - 5b^2c^2}{6(a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2)}$$

$$M_3' = \frac{4c^4 + a^4 + b^4 - 2a^2b^2 - 5a^2c^2 - 5b^2c^2}{6(a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2)}$$

We can therefore determine $d_1 := (OM')^2$, the square of the radius of the disc, to be

$$\frac{a^6 + b^6 + c^6 + 3a^2b^2c^2 - \sum_{sym}(a^4b^2)}{36\sum_{cyc}(2a^2b^2 - a^4)}$$

(this is of course $\frac{1}{4}(OG)^2$, which can be easily verified)

The Mittenpunkt M has normalized coordinates [7]

$$\frac{1}{\sum_{cyc}(2ab - a^2)}(a(b + c - a), b(a + c - b), c(a + b - c))$$

and so we may determine $d_2 := (MM')^2$, the square of the distance from the centre of the disc to the Mittenpunkt, an expression that does not simplify greatly and so is omitted here. If $d_2 \leq d_1$, then, since the distances are non-negative, we can infer $MM' \leq OM'$, which is to say M is no further than the radius of the circle from the centre M' or rather M is constrained to the disc on diameter OG as desired.

Evaluating and simplifying the expression $d_1 - d_2$ gives $-5c^3ba^2 - 4b^2a^4 - 4b^4c^2 - 4c^2a^4 + 18b^2a^2c^2 - 4c^4a^2 - 4b^4a^2 - 4b^2c^4 + a^5b + 6a^3b^3 + ab^5 + 4c^4ab - 5ca^2b^3 - 5ca^3b^2 + 4cb^4a + 4ca^4b - 5c^2a^3b - 5c^3ab^2 + c^5a + c^5b + 6c^3a^3 + 6c^3b^3 + ca^5 + cb^5 - 5c^2ab^3/6(a^2 + b^2 + c^2 - 2ab - 2ac - 2bc)^2$ which, assuming a non-equilateral, non-degenerate triangle, will have strictly positive denominator and hence will be positive (as desired) precisely when the numerator is positive. This unwieldy expression can be more elegantly captured as the homogeneous symmetric inequality

$$\sum_{sym}(3a^2b^2c^2 + a^5b + 3a^3b^3 + 2a^4bc) \geq \sum_{sym}(4a^4b^2 + 5a^3b^2c)$$

Which *Mathematica*[1] confirms is valid for any $a, b, c > 0$. Unfortunately, this inequality cannot be easily proved with Muirhead's Theorem [6] or related results; [5] for instance sets out an algorithm for testing such an inequality which is inconclusive for this example.

[1]Thanks to Stanley Rabinowitz for this calculation

4.2 The Gergonne Point

Simple experimentation reveals that the Gergonne point is neither constrained to, nor restricted from, the Brocard disc. The following figure provides an example of each case.

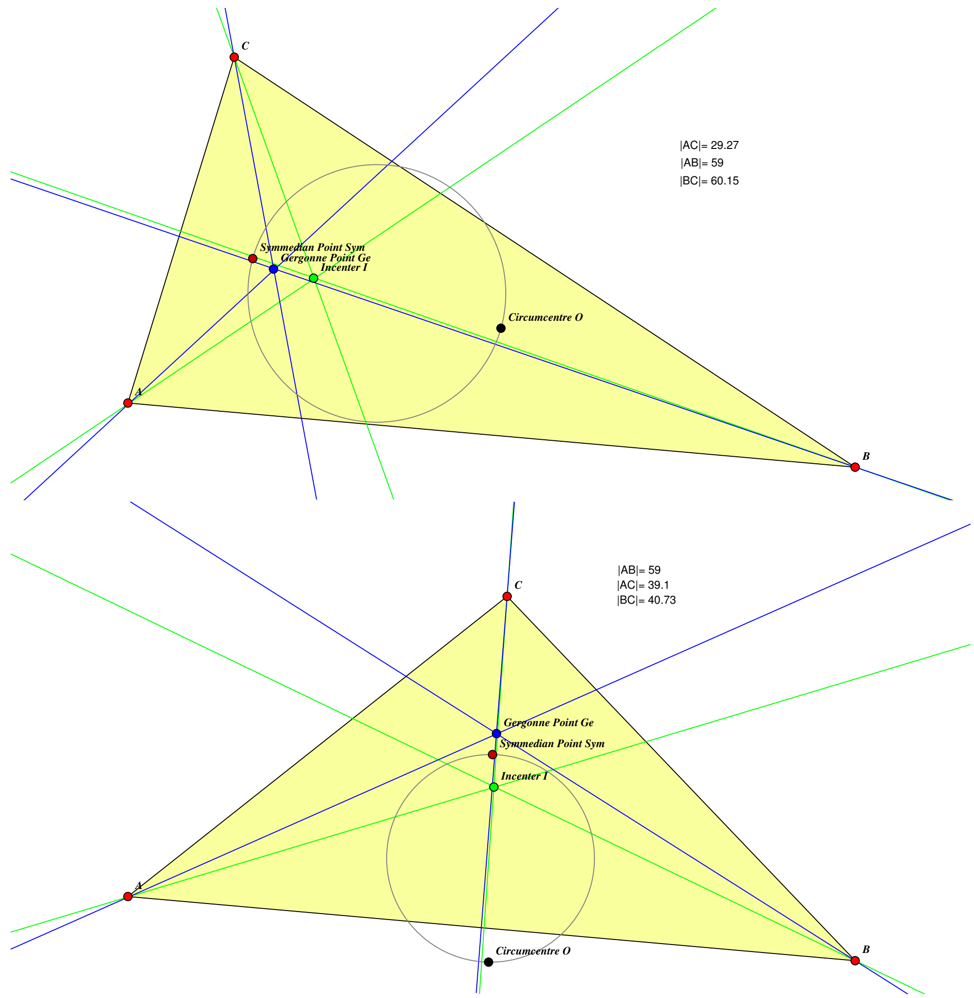

Figure 7: Possible positions of the Gergonne point both in- and outside the Brocard disc.

4.3 The Incenter

The Incenter is conjectured to lie in the intersection of the discs enclosed by the Brocard and Parry circles. Figure 6 illustrates an example.

References

[1] G.C.Smith, "Statics and the moduli space of triangles," *Forum Geometricorum* submitted March 2005.

[2] C.J.Bradley and G.C.Smith, "The locations of triangle centres" *Forum Geometricorum* submitted March 2005.

[3] H.M.Coxeter *Introduction to Geometry*, John Wiley and Sons, 1961.

[4] C.J.Bradley *Challenges in Geometry*, OUP, 2005.

[5] Stanley Rabinowitz, "On the Computer Solution of Symmetric Homogeneous Triangle Inequalities"; in Proceedings of the ACM-SIGSAM 1989 International Symposium on Symbolic and Algebraic Computation (ISAAC 89), July 17-19, 1989. pp. 272-286.

[6] A.W.Marshall and I.Olkin *Inequalities: Theory of Majorization and Its Applications* Academic Press 1979.

[7] Kimberling *The Encyclopedia of Triangle Centers* entry X(9).

[8] All diagrams developed with the Cinderella Interactive Geometry pacakge: *www.cinderella.de*