Lehmer's conjecture for Hermitian matrices over the Eisenstein and Gaussian integers

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Mahler Measure

Let $P(z) = \prod_{i=1}^{d} (z - \alpha_i) \in \mathbb{Z}[z]$ be monic, non-constant.

Definition

The *Mahler Measure* $M(P)$ is given by

$$M(P) = \prod_{i=1}^{d} \max(1, |\alpha_i|)$$

Mahler Measure

- Clearly, $M(P) \geq 1$ for all P.
- $M(P) = 1 \Leftarrow P$ cyclotomic.

Mahler Measure

- Clearly, $M(P) \geq 1$ for all P.
- $M(P) = 1 \Leftrightarrow P$ cyclotomic (ish).

Mahler Measure

- Clearly, $M(P) \geq 1$ for all P.
- $M(P) = 1 \Leftrightarrow$ all nonzero roots of P are roots of unity.

Mahler Measure

Lehmer's Problem

Does there exist $\tau > 1$ such that

$$M(P) > 1 \Rightarrow M(P) \geq \tau?$$

Lehmer's Conjecture, weak version

Yes!

Lehmer's Conjecture, strong version

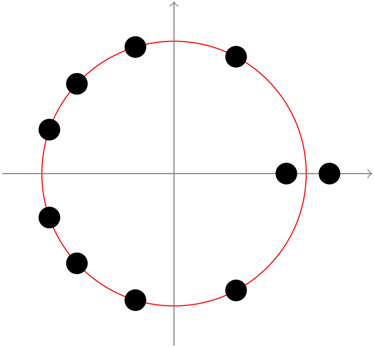
$$M(P) > 1 \Rightarrow M(P) \geq \tau_0 = 1.176280818\ldots$$

Mahler Measure

The best known $M(P) > 1$ was found by DH Lehmer, in 1933:

$$z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

has Mahler measure τ_0.



Associated Polynomials

Let $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, and $M_n(R)$ the ring of $n \times n$ matrices over R.
Let $A \in M_n(R)$ be a Hermitian matrix with characteristic polynomial $\chi_A = det(xI - A)$.

- We define its *associated polynomial* as $R_A(z) := z^n \chi_A(z + 1/z)$
- If A has all eigenvalues in $[-2, 2]$, then $M(R_A) = 1$ (We describe A as a *cyclotomic matrix*.)

Cauchy Interlacing Theorem

Theorem

Let A be a Hermitian $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.
Let B be obtained from A by deleting row i and column i from A.
Then the eigenvalues $\mu_1 \leq \cdots \leq \mu_{n-1}$ of B interlace with those of A: that is,

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n$$

Thus, adding a row and matching column to a cyclotomic matrix either gives a larger cyclotomic matrix, or one that just fails to be.

Equivalence

Let $U_n(R) = \{Q \in M_n(R) \,|\, QQ^* = Q^*Q = I\}$, for Q^* the Hermitian transpose of Q. Conjugation of an $M \in M_n(R)$ by a $Q \in U_n(R)$ preserves the eigenvalues of M, and base ring.

We say that $B \in M_n(R)$ is **strongly equivalent** to $A \in M_n(r)$ if $B = QAQ^*$ or $\overline{QAQ^*}$ for $Q \in U_n(R)$; they are merely **equivalent** if A is strongly equivalent to $\pm B$.

Any cyclotomic A is equivalent to a block diagonal matrix, with each block cyclotomic; if this has more than one block, we call A **decomposable**, else **indecomposable**.

Maximality Cyclotomics & Minimal Noncyclotomics

- A principal submatrix of any cyclotomic matrix is cyclotomic.
- An indecomposable cyclotomic matrix that is not a principal submatrix of any other indecomposable cyclotomic matrix is a **maximal cyclotomic** matrix.
- If A is indecomposable and noncyclotomic, but every induced submatrix is cyclotomic, then A is a **minimal non-cyclotomic** matrix.

Lehmer's Conjecture for various R

Cyclotomic matrices have been classified for various R:

- $R = \mathbb{Z}$ McKee and Smyth, 2007;
- $R = \mathcal{O}_{\mathbb{Q}(d)}$, $d = -2$ and $d < -3$ T. 2010/2011;
- $R = \mathbb{Z}[i]$, $R = \mathbb{Z}[\omega]$ Greaves, 2012;
- $R = \mathcal{O}_{\mathbb{Q}(d)}$, $d > 0$ Greaves, 2012.

Lehmer's Conjecture for various R

For $A \in M_n(R)$ Hermitian, let $M(A) = M(R_A)$. Then it suffices to prove $M(A) \geq \tau_0$ for all minimal non-cyclotomic matrices over a given R. Thus far, this has been done for

- $R = \mathbb{Z}$ McKee and Smyth, 2011;
- $R = \mathcal{O}_{\mathbb{Q}(d)}$, $d = -2$ and $d < -3$ T. 2012;
- $R = \mathcal{O}_{\mathbb{Q}(d)}$, $d > 0$ Greaves, 2012.

In this work, we complete the quadratic extension case by proving Lehmer's conjecture for $R = \mathbb{Z}[i]$ and $R = \mathbb{Z}[\omega]$.

Graphs from Matrices

For $A \in M_n(R)$ We define an R-graph to be a weighted directed graph on vertex set V with weight function $w : V \times V \to R$ given by $w(u, v) = A_{uv}$.

Vertices u, v are adjacent if the **edge weight** $w(u, v) \neq 0$; vertices can themselves have weight if $w(u, u) \neq 0$, in which case we call vertex u **charged**.

Equivalence of matrices gives equivalence of graphs under the operations of permuting vertex labels; **switching** at a vertex by unit μ; negating all edge weights and charges; or taking complex conjugates of same.

Graphs from Matrices

We may restrict out attention to a finite set of weights.

Charges

Uncharged ●; Positive charge c ; Negative charge $-c$ .

$\mathbb{Z}[i]$-graph edges

1

-1

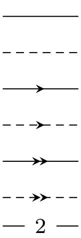
i

$-i$

$1+i$

$-1-i$

2



$\mathbb{Z}[\omega]$-graph edges

1

-1

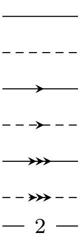
ω

$-\omega$

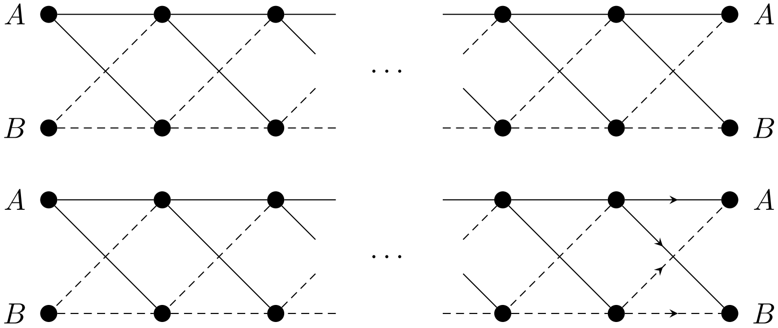
$1+\omega$

$-1-\omega$

2



Infinite Families of Cyclotomics



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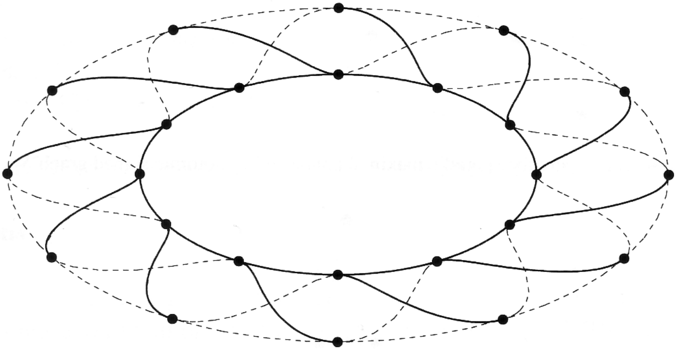
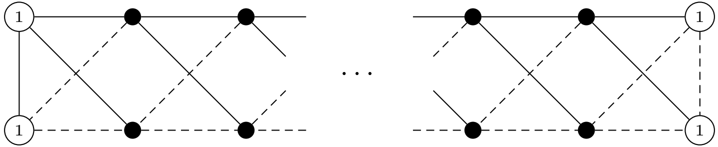
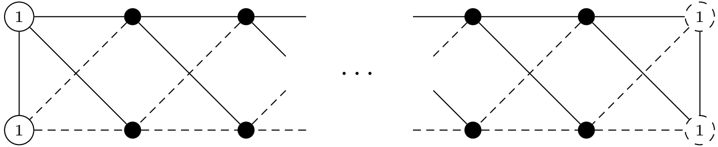
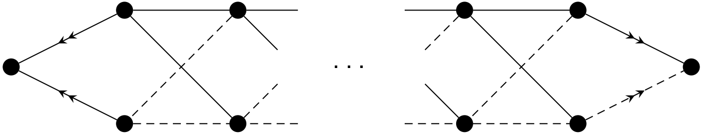


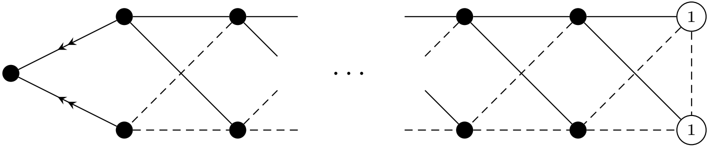
Infinite Families of Cyclotomics





Infinite Families of Cyclotomics





Sporadics

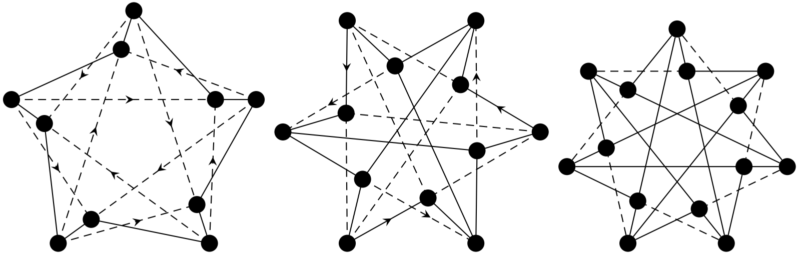


Figure: The sporadic maximal connected cyclotomic $\mathbb{Z}[\omega]$-graphs S_{10}, S_{12}, and S_{14} of orders 10, 12, and 14 respectively. The $\mathbb{Z}$-graph S_{14} is also a $\mathbb{Z}[i]$-graph.

Sporadics

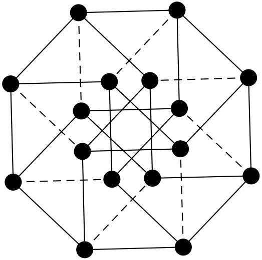


Figure: The sporadic maximal connected cyclotomic $\mathbb{Z}$-hypercube S_{16}.

Sporadics

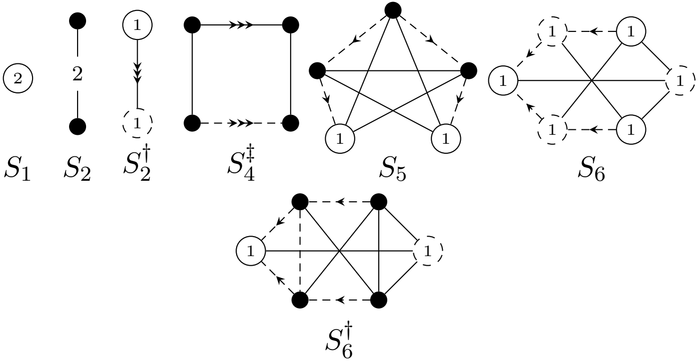


Figure: The sporadic maximal connected cyclotomic $\mathbb{Z}[\omega]$-graphs of orders 1, 2, 4, 5, and 6. The $\mathbb{Z}$-graphs S_1 and S_2 are also $\mathbb{Z}[i]$-graphs.

Sporadics

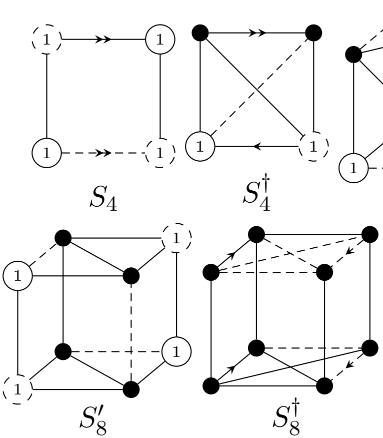


Figure: The sporadic maximal connected cyclotomic $\mathbb{Z}[i]$-graphs of orders 4, 7, and 8. The $\mathbb{Z}$-graphs S_7, S_8, and S_8' are also $\mathbb{Z}[\omega]$-graphs.

Minimal Non-cyclotomics

If G is minimal non-cyclotomic, then every induced subgraph is equivalent to a subgraph of one of the above. If any of its subgraphs is a subgraph of a sporadic but not of one of the non-sporadics, we call G **supersporadic**.

There can only be finitely many supersporadics, with at most 17 vertices.

Non-supersporadics

There are potentially infinitely many of these!
However, for sufficiently large graphs, having every proper subgraph cyclotomic forces cyclotomicity:

Proposition

Let G be a connected non-supersporadic $\mathbb{Z}[i]$-graph (or $\mathbb{Z}[\omega]$-graph) with $n \geqslant 10$ vertices. Then G is equivalent to a subgraph of a non-sporadic $\mathbb{Z}[i]$-graph (or $\mathbb{Z}[\omega]$-graph).

Finite Search

We therefore have two searches:

- Minimal non-cyclotomics of at most ten vertices;
- Supersporadic minimal non-cyclotomics of eleven-seventeen vertices.

The latter are necessarily supergraphs of ten-sixteen vertex subgraphs of S_{10}, S_{12}, S_{14} or S_{16}, and can be brute-forced in this way. The smaller examples we build up from single vertex 'seeds' by interlacing.

Finite Search

We accelerate the search by noting two types of excludable subgraphs:

- Type-I graphs are n-vertex non-cyclotomic graphs; they can be tested for minimality, then excluded for any candidate m-vertex minimal non-cyclotomic graph with $m \geq n$;
- Type-II are cyclotomic, but only appear in finitely many maximal cyclotomics. If the largest such has n vertices, then a minimal non-cyclotomic inducing the subgraph has at most $n + 1$ vertices.

We also reduce mod equivalence for as long as possible - and use a lot of cpuhours!

Lehmer's problem

Theorem (T.,2012)

For $d < 0$, $d \neq -1, -3$: If G is a minimal noncyclotomic R-graph with $M(G) < 1.3$, then G is equivalent to a $\mathbb{Z}$-graph.

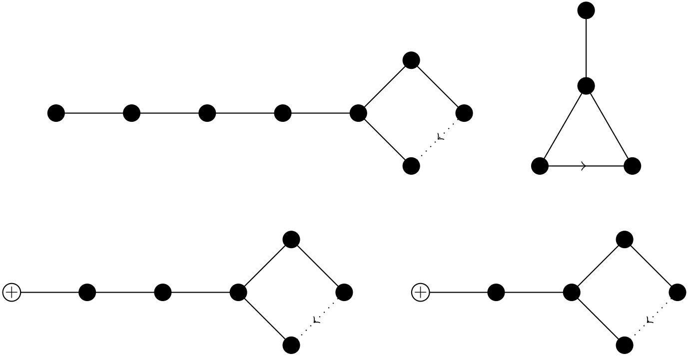
Theorem

If G is a minimal noncyclotomic $\mathbb{Z}[i]$-graph with $M(G) < 1.3$, then G is equivalent to a $\mathbb{Z}$-graph.

Theorem

If G is a minimal noncyclotomic $\mathbb{Z}[\omega]$-graph with $M(A) < 1.3$, then G is equivalent to a charged signed graph or one of four $\mathbb{Z}[\omega]$-graphs, which have Mahler measure at least $1.267\ldots$.

$\mathbb{Z}[\omega]$-graphs with $M(G) < 1.3$



Lehmer's Problem

Corollary

Let A be an R-matrix, for $R = \mathcal{O}_{\mathbb{Q}\sqrt{d}}$. Then

$$M(A) = 1 \text{ or } M(A) \geq \tau_0$$

However:

- The new associated reciprocal polynomials of minimal non-cyclotomic $\mathbb{Z}[\omega]$-graphs were already known;
- There are still 'missing' polynomials, so Lehmer's problem remains open!

Further Reading

`http://maths.straylight.co.uk`

arXiv preprint of this work: 1206.1734.