Modern Cryptography

Public Key Systems for Secret Sharing

Graeme Taylor

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What is Cryptography?

Cryptography

From Greek *kryptos* - hidden - and *graphos* - writing - cryptography is the use of codes to disguise messages.

The main challenge in cryptography

Cryptography

How can you communicate securely over an insecure channel?

Classical Cryptography

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For instance, with a shift of 4:

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Cipher	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D

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The word "cryptography" would thus become "GVCTSKVETLC". The shift number 4 is the "key" to both locking and unlocking the *enciphered* message (symmetric encryption).

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- The receiver converts the ciphertext back into *plaintext* using a corresponding decryption system.

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Classical Cryptography

Problems with classical cryptography

- If an adversary learns the decryption key and system, they can decipher messages, and thus secrecy is lost.
- If an adversary learns the encryption key and system, they can encipher messages, and thus trust is lost.

Classical Cryptography

The biggest problem with private key cryptography

In order to share secrets, you must first have shared a secret!

Public Key Cryptography

The challenge for modern cryptography

Can you establish a secret with a previously uncontacted stranger, without sharing the same secret with anyone listening in?

Secret sharing with paint

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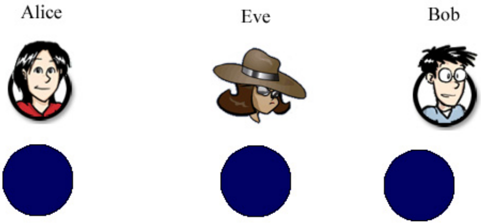

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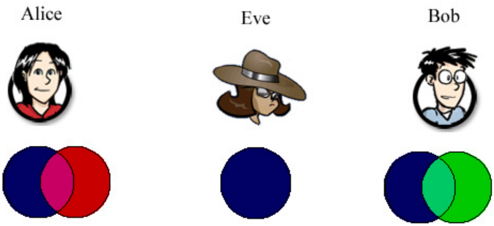

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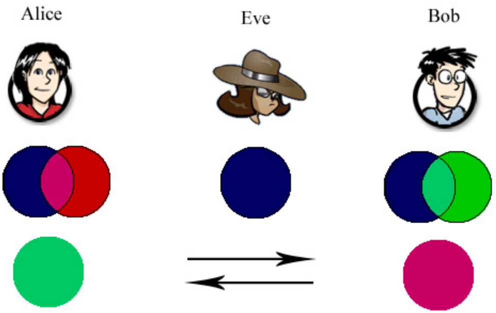

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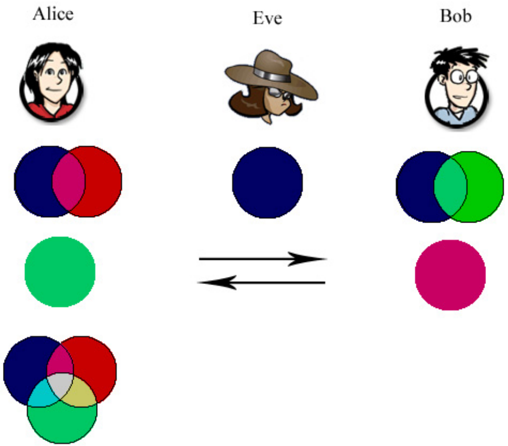

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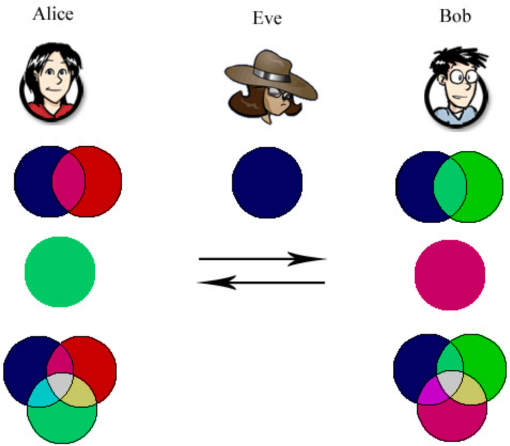

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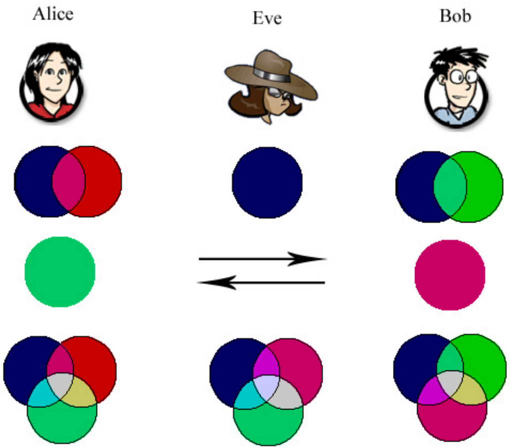

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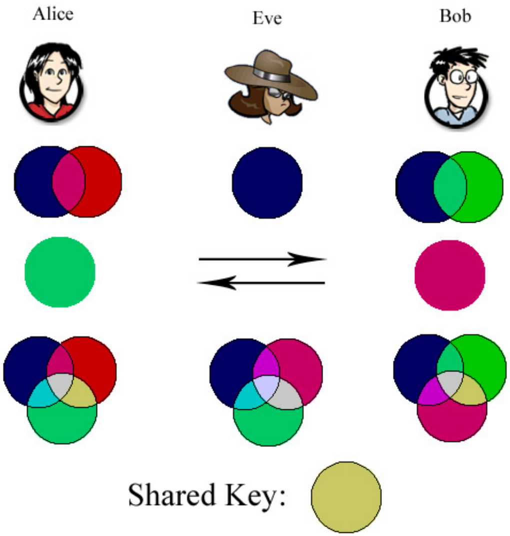

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- But you can't unmix paint, so Eve can't learn a private colour from the mix and the base.
- So Eve can't learn the shared secret.
- The order in which paint is mixed does not matter- so Alice and Bob reach the same secret result.

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Definition

An injective function is described as a *one-way function* if, like mixing paint, it's easy to compute the output from the inputs, but (practically) impossible to compute the inputs from the output.

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Problem

No one has managed to prove that a one-way function really exists!

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So we can consider the map

$$f : \mathbb{Z}/N\,\mathbb{Z} \times G \to G$$

$$f(n, g) = [n]g$$

Computing f

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Then $T = [t]g = h$ as required.

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If $\mathcal{A}$ is an algorithm that reads in g, h, performs m group operations and then returns an answer $v \in \mathbb{Z}/N\mathbb{Z}$, then the probability that $t = v$ is $O(m^2/p)$, for p the largest prime dividing N.

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So for a non-negligible probability of success, $\mathcal{A}$ must perform $O(\sqrt{p})$ group operations.

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- So we can use prime groups to securely generate shared secrets.

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Currently fashionable choice is the group of rational points of an elliptic curve over a finite field, since there is no obvious reduction to $\mathbb{Z}/p\mathbb{Z}$.

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- Even with perfect crypto and perfect protocols, implementation may disclose secrets.
- After establishing a secret, we need a classical cryptosystem that's at least as secure.

XLEROCSY JSV PMWXIRMRK!

THANKYOU FOR LISTENING!