Integer Matrices with Constrained Eigenvalues

Cyclotomic matrices and charged signed graphs

Graeme Taylor

Edinburgh

January 2009

A question

Which integer symmetric matrices have all eigenvalues in $[-2, 2]$?

Mahler Measure

Let $P(z) = a_0 z^d + \cdots + a_d = a_0 \prod_{i=1}^{d}(z - \alpha_i)$ be a non-constant polynomial.

Mahler Measure

Let $P(z) = a_0 z^d + \cdots + a_d = a_0 \prod_{i=1}^{d}(z - \alpha_i)$ be a non-constant polynomial.

Definition

The *Mahler Measure* $M(P)$ is given by

$$M(P) := |a_0| \prod_{i=1}^{d} \max\left(1, |\alpha_i|\right)$$

Mahler Measure

Let $P(z) = a_0 z^d + \cdots + a_d = a_0 \prod_{i=1}^{d}(z - \alpha_i)$ be a non-constant polynomial.

Definition

The *Mahler Measure* $M(P)$ is given by

$$M(P) := |a_0| \prod_{i=1}^{d} \max(1, |\alpha_i|)$$

- Clearly, $M(P) \geq 1$ for all P.

Mahler Measure

Let $P(z) = a_0 z^d + \cdots + a_d = a_0 \prod_{i=1}^{d}(z - \alpha_i)$ be a non-constant polynomial.

Definition

The *Mahler Measure* $M(P)$ is given by

$$M(P) := |a_0| \prod_{i=1}^{d} \max(1, |\alpha_i|)$$

- Clearly, $M(P) \geq 1$ for all P.
- If $M(P) = 1$, then all roots of P lie in the unit circle.

Mahler Measure

- If A is an $n \times n$ integer symmetric matrix, then its *associated polynomial* is $R_A(z) := z^n \chi_A(z + 1/z)$

Mahler Measure

- If A is an $n \times n$ integer symmetric matrix, then its *associated polynomial* is $R_A(z) := z^n \chi_A(z + 1/z)$
- If A has all eigenvalues in $[-2, 2]$, then R_A is a cyclotomic polynomial- We describe A as a *cyclotomic matrix*.

Mahler Measure

- If A is an $n \times n$ integer symmetric matrix, then its *associated polynomial* is $R_A(z) := z^n \chi_A(z + 1/z)$
- If A has all eigenvalues in $[-2, 2]$, then R_A is a cyclotomic polynomial- We describe A as a *cyclotomic matrix*.
- So cyclotomic matrices yield integer polynomials R_A with the minimal possible Mahler measure!

Mahler Measure

But *any* cyclotomic polynomial will have Mahler measure 1- why bother with the intermediate step of cyclotomic matrices?

Lehmer's Conjecture

Now suppose P is a monic polynomial with integer coefficients.

- Lehmer's Problem: For such polynomials with $M(P) > 1$, can $M(P)$ be arbitrarily close to 1?

Lehmer's Conjecture

Now suppose P is a monic polynomial with integer coefficients.

- Lehmer's Problem: For such polynomials with $M(P) > 1$, can $M(P)$ be arbitrarily close to 1?
- If not, then there exists some $\lambda > 1$ such that $M(P) > 1 \Rightarrow M(P) > \lambda$, forcing a 'gap' between cyclotomic and non-cyclotomic polynomials.

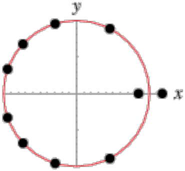
Lehmer's Conjecture

The smallest known Mahler measure greater than 1 for a monic polynomial from $\mathbb{Z}[z]$ is

$$\lambda_0 = 1.176280818$$

which is the larger real root of the *Lehmer polynomial*

$$z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$



From cyclotomic to non-cyclotomic?

- Likely candidates for small Mahler measure are polynomials that are ‘almost cyclotomic’- as few roots outside the unit circle as possible.

From cyclotomic to non-cyclotomic?

- Likely candidates for small Mahler measure are polynomials that are 'almost cyclotomic'- as few roots outside the unit circle as possible.
- Difficulty: There's no obvious way to obtain such an 'almost cyclotomic' integer polynomial from a cyclotomic one.

From cyclotomic to non-cyclotomic?

- Likely candidates for small Mahler measure are polynomials that are ‘almost cyclotomic’- as few roots outside the unit circle as possible.
- Difficulty: There’s no obvious way to obtain such an ‘almost cyclotomic’ integer polynomial from a cyclotomic one.
- But given a cyclotomic matrix, we can tweak it slightly to give a non-cyclotomic matrix.

From cyclotomic to non-cyclotomic

Theorem (Cauchy Interlacing Theorem)

Let A be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.
Let B be obtained from A by deleting row i and column i from A. Then the eigenvalues $\mu_1 \leq \cdots \leq \mu_{n-1}$ of B interlace with those of A: that is,

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n$$

From cyclotomic to non-cyclotomic

We can run this process in reverse. Let B be a cyclotomic matrix, so its eigenvalues satisfy

$$-2 \leq \mu_1 \leq \cdots \leq \mu_{n-1} \leq 2$$

From cyclotomic to non-cyclotomic

We can run this process in reverse. Let B be a cyclotomic matrix, so its eigenvalues satisfy

$$-2 \leq \mu_1 \leq \cdots \leq \mu_{n-1} \leq 2$$

Then if we 'grow' a matrix A from B by adding an extra row and column, we have by interlacing

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n$$

From cyclotomic to non-cyclotomic

We can run this process in reverse. Let B be a cyclotomic matrix, so its eigenvalues satisfy

$$-2 \leq \mu_1 \leq \cdots \leq \mu_{n-1} \leq 2$$

Then if we 'grow' a matrix A from B by adding an extra row and column, we have by interlacing

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n$$

So

$$\lambda_2, \ldots, \lambda_{n-1} \in [\mu_1, \mu_{n-1}] \subseteq [-2, 2]$$

From cyclotomic to non-cyclotomic

We can run this process in reverse. Let B be a cyclotomic matrix, so its eigenvalues satisfy

$$-2 \leq \mu_1 \leq \cdots \leq \mu_{n-1} \leq 2$$

Then if we 'grow' a matrix A from B by adding an extra row and column, we have by interlacing

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n$$

So

$$\lambda_2, \ldots, \lambda_{n-1} \in [\mu_1, \mu_{n-1}] \subseteq [-2, 2]$$

At worst,

$$\lambda_1, \lambda_n \notin [-2, 2]$$

Cyclotomic Matrices: Indecomposability

If M decomposes as a block-diagonal matrix, then its eigenvalues are those of the blocks. So we can build cyclotomic matrices from blocks of smaller ones; to classify cyclotomic matrices it therefore suffices to classify just the indecomposable ones.

Cyclotomic Matrices: Interlacing I

- If A is cyclotomic, so is any B obtained by deleting some set of rows and corresponding columns of A: B is described as being *contained in A*.

Cyclotomic Matrices: Interlacing I

- If A is cyclotomic, so is any B obtained by deleting some set of rows and corresponding columns of A: B is described as being *contained in* A.
- If M is an indecomposable cyclotomic matrix that is not contained in any strictly larger indecomposable cyclotomic matrix, then M is described as being *maximal*.

Cyclotomic Matrices: Interlacing I

- If A is cyclotomic, so is any B obtained by deleting some set of rows and corresponding columns of A: B is described as being *contained in A*.
- If M is an indecomposable cyclotomic matrix that is not contained in any strictly larger indecomposable cyclotomic matrix, then M is described as being *maximal*.
- Theorem: Any non-maximal indecomposable cyclotomic matrix is contained in a maximal one.

Cyclotomic Matrices: Equivalence

Let $O_n(\mathbb{Z})$ be the orthogonal group of $n \times n$ signed permutation matrices, generated by matrices of the form $diag(1, 1, \ldots, 1, -1, 1, \ldots, 1)$ and permutation matrices.

Cyclotomic Matrices: Equivalence

Let $O_n(\mathbb{Z})$ be the orthogonal group of $n \times n$ signed permutation matrices, generated by matrices of the form $diag(1, 1, \ldots, 1, -1, 1, \ldots, 1)$ and permutation matrices.

- If M is cyclotomic and $P \in O_n(\mathbb{Z})$, then $M' = PMP^{-1}$ is cyclotomic since it has the same eigenvalues. We describe M and M' as *strongly equivalent*.

Cyclotomic Matrices: Equivalence

Let $O_n(\mathbb{Z})$ be the orthogonal group of $n \times n$ signed permutation matrices, generated by matrices of the form $diag(1, 1, \ldots, 1, -1, 1, \ldots, 1)$ and permutation matrices.

- If M is cyclotomic and $P \in O_n(\mathbb{Z})$, then $M' = PMP^{-1}$ is cyclotomic since it has the same eigenvalues. We describe M and M' as *strongly equivalent*.
- A matrix M' is then described as *equivalent* to M if it is strongly equivalent to either M or $-M$.

Cyclotomic Matrices: Interlacing II

Lemma

The only cyclotomic 1×1 *matrices are*

$$(0), (1), (-1), (2), (-2)$$

Cyclotomic Matrices: Interlacing II

Lemma

The only cyclotomic 1×1 *matrices are*

$$(0), (1), (-1), (2), (-2)$$

Corollary

By interlacing, the entries of an integer cyclotomic matrix must be elements of $\{0, 1, -1, 2, -2\}$.

Cyclotomic Matrices: Interlacing II

Lemma

The only cyclotomic 1×1 *matrices are*

$$(0), (1), (-1), (2), (-2)$$

Corollary

By interlacing, the entries of an integer cyclotomic matrix must be elements of $\{0, 1, -1, 2, -2\}$.

Lemma

Apart from matrices equivalent to (2) *or* $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, *any indecomposable cyclotomic matrix has all entries from the set* $\{0, 1, -1\}$.

The question, refined

Our original question thus reduces to classifying all *maximal, indecomposable, cyclotomic, symmetric* $\{-1,0,1\}$*-matrices, up to equivalence*.

Charged Signed Graphs

A convenient representation of such a matrix M is given by a *charged, signed graph* G.

Charged Signed Graphs

A convenient representation of such a matrix M is given by a *charged, signed graph* G.

- $M_{ii} = 0$ gives a neutral vertex i, denoted $\bullet$.
- $M_{ii} = 1$ gives a positively-charged vertex i, denoted $\oplus$.
- $M_{ii} = -1$ gives a negatively-charged vertex i, denoted $\ominus$.

Charged Signed Graphs

A convenient representation of such a matrix M is given by a *charged, signed graph* G.

- $M_{ii} = 0$ gives a neutral vertex i, denoted $\bullet$.
- $M_{ii} = 1$ gives a positively-charged vertex i, denoted $\oplus$.
- $M_{ii} = -1$ gives a negatively-charged vertex i, denoted $\ominus$.
- $M_{ij} = 1$, $i \neq j$ gives a positive edge between vertices i and j, denoted ——— .
- $M_{ij} = -1$, $i \neq j$ gives a negative edge between vertices i and j, denoted $\cdots\cdots$.

Cyclotomic Graphs

- M indecomposable $\Leftrightarrow$ G connected.

Cyclotomic Graphs

- M indecomposable $\Leftrightarrow$ G connected.
- Maximality: M not contained in a larger cyclotomic matrix $\Leftrightarrow$ G not an induced subgraph of a larger cyclotomic graph.

Cyclotomic Graphs

- M indecomposable $\Leftrightarrow$ G connected.
- Maximality: M not contained in a larger cyclotomic matrix $\Leftrightarrow$ G not an induced subgraph of a larger cyclotomic graph.
- M_1 a permutation of M_2 $\Leftrightarrow$ G_1 is a re-labelling of G_2.

Cyclotomic Graphs

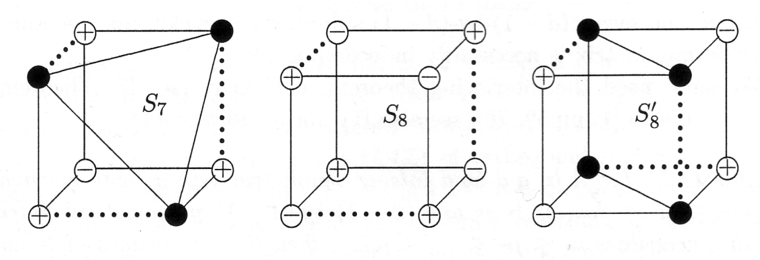
- M indecomposable $\Leftrightarrow$ G connected.
- Maximality: M not contained in a larger cyclotomic matrix $\Leftrightarrow$ G not an induced subgraph of a larger cyclotomic graph.
- M_1 a permutation of M_2 $\Leftrightarrow$ G_1 is a re-labelling of G_2.
- Conjugation of M by kth diagonal matrix $\Leftrightarrow$ Switching of signs of all edges incident at vertex k of G.

A picture is worth a thousand matrices

So we can represent an equivalence class of cyclotomic matrices by a cyclotomic graph.

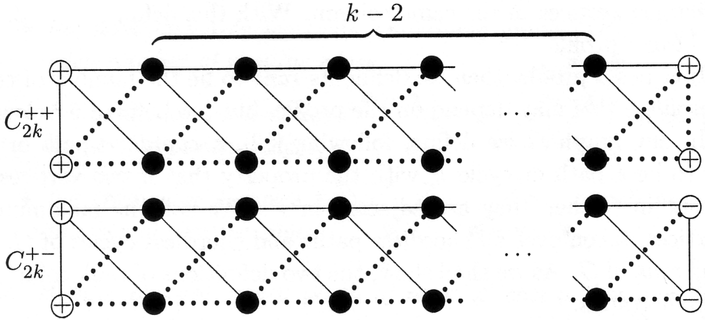
Classification

Charged Sporadics S_7, S_8, S_8':



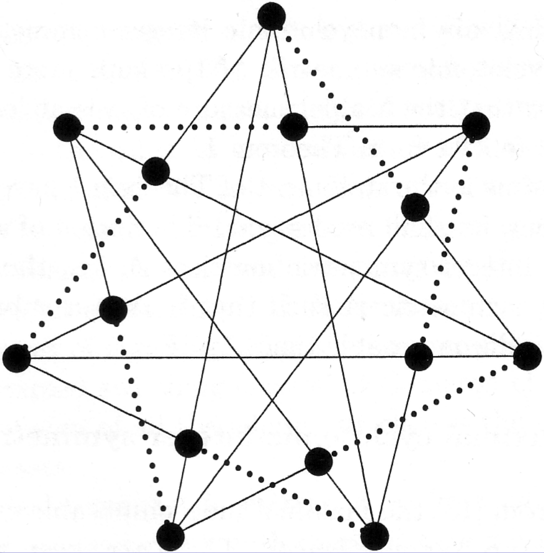
Classification

Infinite family $C_{2k}^{+\pm}$, $k \geq 2$:



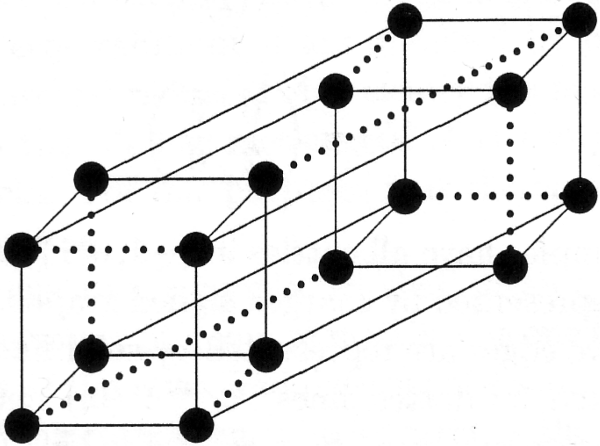
Classification

Uncharged Sporadic S_{14}:



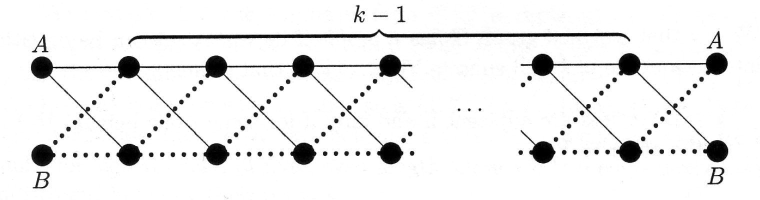
Classification

Uncharged Sporadic S_{16}:

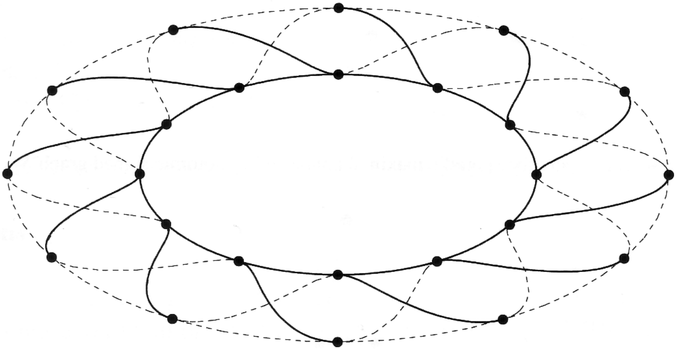


Classification

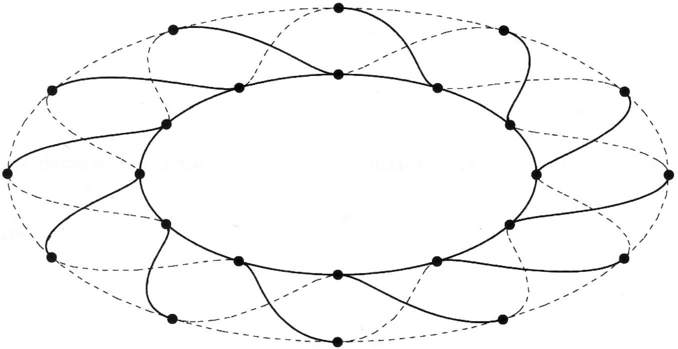
Infinite family T_{2k}, $k \geq 3$:



Thanks for listening!



Thanks for listening!



Slides and references online at http://maths.straylight.co.uk