Topics in Discrete Mathematics
Introduction to Graph Theory

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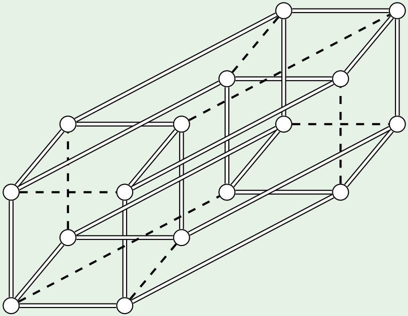
5/ii/13

Definition

A **signed graph** is a graph $G = (V, E)$ together with a sign function $s : E \to \{+1, -1\}$. We say that an edge $e \in E$ is **positive** if $s(e) = 1$, and **negative** if $s(e) = -1$.

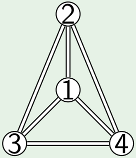
We will indicate the sign function visually by drawing negative edges with dashed lines:

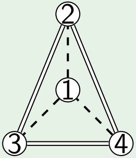
Example



- The adjacency matrix is now a symmetric $\{0, 1, -1\}$-matrix with 0 diagonal.
- Two signed graphs G, G' with adjacency matrices $\mathbf{A}$, $\mathbf{A}'$ are **equivalent** if there exists a **signed permutation matrix Q** such that $\mathbf{A}' = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$.
- Such a $\mathbf{Q}$ decomposes as $\mathbf{Q} = \mathbf{P}\mathbf{S}$ where $\mathbf{P}$ is a permutation matrix and $\mathbf{S}$ is a **switching matrix**.
- $\mathbf{S}$ is a diagonal matrix with each $S_{ii} = \pm 1$; if $S_{ii} = -1$ then $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ swaps all the signs of edges incident at vertex i in the signed graph with adjacency matrix $\mathbf{A}$.

Example

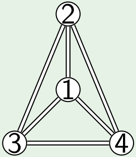


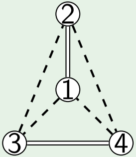


$$\underbrace{\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}}_{\mathbf{A_1}} \qquad \underbrace{\begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix}}_{\mathbf{A_2}}$$

$$\mathbf{A_2} = \mathbf{S}^{-1}\mathbf{A_1}\mathbf{S} \text{ for } \mathbf{S} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Example

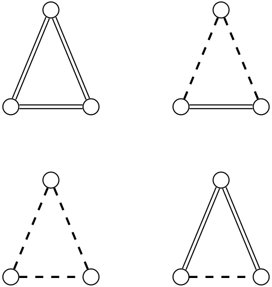




$$\mathbf{A_1} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad \mathbf{A_3} = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$

$$\mathbf{A_3} = \mathbf{S}'^{-1}\mathbf{A_1}\mathbf{S}' \text{ for } \mathbf{S}' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The possible signed triangles are



We call the first two **balanced**: they capture mutual friendship, and the proverb “the enemy of my enemy is my friend”.
The second two are **unbalanced**, as they introduce tension.

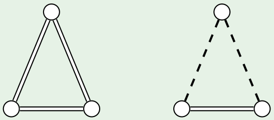
Definition

A **complete signed graph** $G = (V, E)$ is one in which we have a signed (positive or negative) edge between *every* pair of vertices $v, w \in V$.

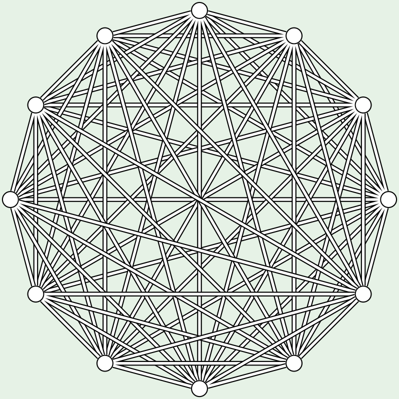
Definition

A complete signed graph G is called **structurally balanced** if every triangle in G is balanced.

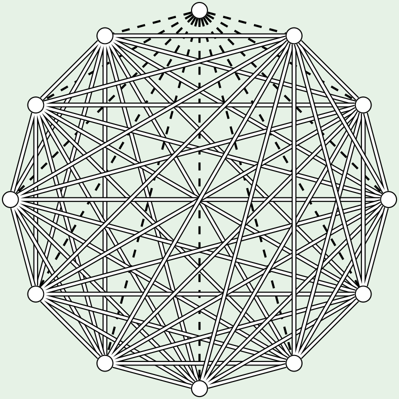
Example



Example



Example



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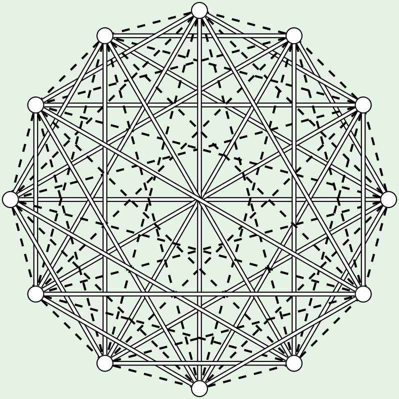

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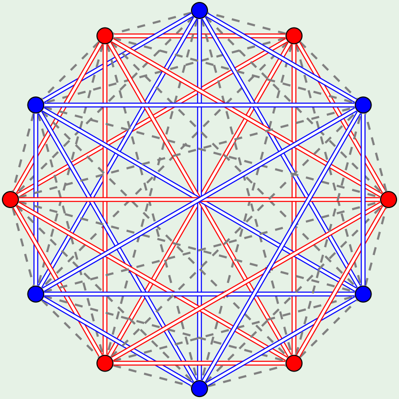
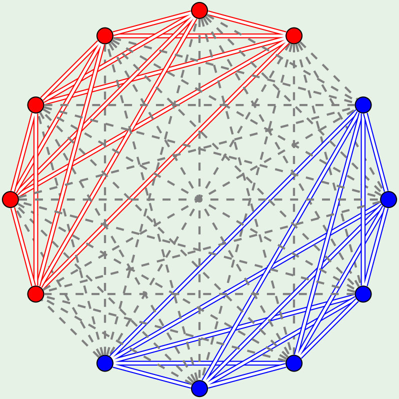


Example



Theorem (Complete Structural Balance)

A complete signed graph $G = (V, E)$ is structurally balanced

if and only if

$V = A \cup B$ for (possibly empty) disjoint sets A, B such that an edge of G is negative if and only if one endpoint is in A and the other is in B.

$\Rightarrow$:

- Pick some vertex u in V. Define A to be every vertex v such that $s(e_{uv}) = 1$, and B to be the rest; as G is complete, every vertex is therefore in one of A and B, but only one.
- Pick any two vertices v, w in A. Then $s(e_{uv}) = 1 = s(e_{uw})$ by construction; by structural balance, the triangle on u, v, w cannot have a single negative edge, so $s(e_{vw}) = 1$.
- Pick any two vertices v, w in B. Then $s(e_{uv}) = -1 = s(e_{uw})$ by construction; by structural balance, the triangle on u, v, w cannot have three negative edges, so $s(e_{vw}) = 1$.
- Finally, pick $v \in A$, $w \in B$. Then $s(e_{uv}) = 1$ and $s(e_{uw}) = -1$ by construction; by structural balance, the triangle on u, v, w cannot have a single negative edge, so $s(e_{vw}) = -1$.

$\Leftarrow$: Suppose we have a balanced partition into sets A, B. Let T be a triangle in G. Then either:

- All three vertices of T are in A (so all three edges of T are positive);
- Or all three vertices of T are in B (so all three edges of T are positive);
- Or two vertices u, v are in one of A, B, and one (w) is in the other. Then e_{uv} is positive and e_{uw}, e_{vw} are negative, so T is still a balanced triangle.

The complete structural balance theorem shows that a local property (each triangle balanced) is equivalent to a global property (existence of a balanced partition).

Definition

Generalisation 1: We define *any* signed graph to be structurally balanced if we can 'complete' it by filling in the missing edges without creating an unbalanced triangle. (i.e., it is a subgraph of a structurally balanced complete signed graph).

Definition

Generalisation 2: We define *any* signed graph to be structurally balanced if its vertices can be partitioned into two disjoint sets A, B such that any edge with both endpoints in A, or both endpoints in B, is positive; whilst any edge with one endpoint in each set is negative.

Proposition

These rival definitions are equivalent!

$\Rightarrow$ If G is a subgraph of a complete signed graph G' that is structurally balanced, then we can use the partition of G' as a partition of G; since G' has no positive edges crossing from A to B, nor negative edges within A or B, G also cannot.

$\Leftarrow$ Given a balanced partition of an incomplete signed graph $G = (V, E)$, for any $u, v \in V = A \cup B$ such that $e_{uv} \notin E$, we simply set the sign of e_{uv} to be 1 if $u, v \in A$ or $u, v \in B$, or -1 otherwise. This provides a completion of the graph to a structurally balanced complete signed graph.

Proposition

Let G be a structurally balanced signed graph. Then any cycle in G has an even number of negative edges.

- Let C be a cycle starting at ending at some v, and S whichever of A, B contains v; w.l.o.g., assume $S = A$.
- Walk along C from v, switching S between A and B each time a negative edge is encountered; thus S always tracks the set containing the current vertex.
- If there are an odd number of negative edges, we'll swap an odd number of times, so the walk will end with $S = B$. But the walk ends at $v \in A$, a contradiction.

Theorem

Let G be a signed graph. Then G is structurally balanced if and only if every cycle in G has an even number of negative edges.

Proposition

A signed graph is structurally balanced if and only if all its connected components are.

Balanced Partition Algorithm

Let $G = (V, E)$ be a connected signed graph in which every cycle has an even number of negative edges. Pick any $u \in V$, and set $A = \{u\}, B = \emptyset$. We will populate A, B as follows.

1. If every vertex is in A or B, terminate.
2. Otherwise, there exists at least one v which has not been assigned to a set, with a neighbour w that has. Set the *predecessor* $p(v)$ to be w.
3. - If $w \in A$ and $s(e_{vw}) = 1$, assign v to A.
   - If $w \in B$ and $s(e_{vw}) = 1$, assign v to B.
   - If $w \in A$ and $s(e_{vw}) = -1$, assign v to B.
   - If $w \in B$ and $s(e_{vw}) = -1$, assign v to A.
4. Return to step 1.

By the end of the algorithm, every vertex v has been assigned to exactly one of A, B, and has been given a predecessor $p(v)$. We need to show that A, B is a balanced partition.

Proposition

For a $v \in V$, let P_v be the *path of predecessors*

$$v \to p(v) = v_1 \to p(v_1) = v_2 \to \cdots \to p(v_i) = u.$$

Then P_v has an even number of negative edges if and only if $v \in A$.

Proposition

Let $v \in A$, and w be any vertex adjacent to v such that $s(e_{vw}) = 1$. Then (as required), $w \in A$.

- Let $P_v = v, v_1, \ldots, v_i = u$ and $P_w = w, w_1, \ldots, w_j = u$ be the respective paths of predecessors of v and w. Consider the cycle
$$C_{vw} : v, v_1, \ldots, v_i, u, w_j, \ldots, w, v.$$
- So $C_{vw} = P_v \circ P_w^{-1} \circ e_{wv}$, where P_w^{-1} is the reversal of P_w. As a cycle in G, C_{vw} has an even number of negative edges.
- The subpath P_v has an even number of negative edges since $v \in A$, and e_{wv} is positive by assumption, so P_w^{-1} must also have an even number of negative edges. Therefore P_w does, so $w \in A$.

Proposition

Let $v \in B$, and w be any vertex adjacent to v such that $s(e_{vw}) = 1$. Then (as required), $w \in B$.

Now P_v has an odd number of negative edges because v is in a different set to u. Since e_{vw} is still positive, this forces P_w^{-1} to have an odd number of negative edges to ensure C_{vw} has an even number of such edges. So $w \in B$.

Proposition

Let $v \in A$, and w be any vertex adjacent to v such that $s(e_{vw}) = -1$. Then (as required), $w \in B$.

Now P_v has an even number of negative edges, but e_{vw} is negative, so P_w^{-1} must have an odd number of negative edges to balance C_{vw}. So $w \in B$.

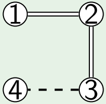
Proposition

Let $v \in B$, and w be any vertex adjacent to v such that $s(e_{vw}) = -1$. Then (as required), $w \in A$.

Now P_v has an odd number of negative edges, and e_{vw} is also negative, so P_w^{-1} must have an even number of negative edges to balance C_{vw}. So $w \in A$.

Example

In a structurally balanced network,



The enemy of the friend of my friend is...my enemy

Example

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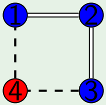

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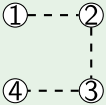

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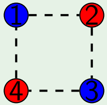

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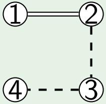

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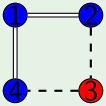
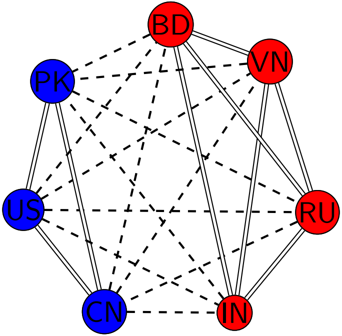


The enemy of the enemy of my friend is...my friend

In 1972, Bangladesh separated from Pakistan; the following explanation of how international support divided appeals to arguments of structural balance:

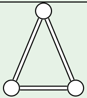
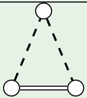
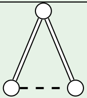
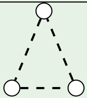
> *"The United States's somewhat surprising support of Pakistan ... becomes less surprising when one considers that the USSR was China's enemy, China was India's foe, and India had traditionally bad relations with Pakistan. Since the U.S. was at that time improving its relations with China, it supported the enemies of China's enemies. Further reverberations of this strange political constellation became inevitable: North Vietnam made friendly gestures toward India... and China vetoed the acceptance of Bangladesh into the U.N."*



BD
VN
PK
US
RU
CN
IN

Example

Player interactions in an MMO.

				
Stability	stable	stable	unstable	unstable
N_Δ, observed	26,329	39,519	4,428	8,032
N_Δ, random model	10,608	28,545	30,145	9,009

Theorem (Approximate Balance)

Let $0 \leq \epsilon < 1/8$, and $\delta = \sqrt[3]{\epsilon}$. If $G = (V, E)$ is a complete signed graph in which at least $1 - \epsilon$ of all triangles are balanced, then either:

1. *There is a set S consisting of at least $1 - \delta$ of the vertices, in which at least $1 - \delta$ of all edges between elements of S are positive;*
2. *Or V can be partitioned into sets A, B such that*
   - *at least $1 - \delta$ of the edges in A are positive;*
   - *at least $1 - \delta$ of the edges in B are positive;*
   - *At least $1 - \delta$ of the edges with one endpoint in A and the other in B are negative.*

Picking a 'good' vertex

- An n-vertex complete graph has $n(n-1)/2$ edges and $n(n-1)(n-2)/6$ triangles.
- Let ϵ be the fraction of unbalanced triangles, and U the total number. So $U = \epsilon n(n-1)(n-2)/6$.
- Let the *defect* $d(v)$ be the number of unbalanced triangles that have v as a vertex.
- So $\sum_{v \in V} d(v) = 3U$, and the average vertex has defect $3U/n$.
- Not all vertices can have greater than average defect, so there is at least one u with $d(u) \leq 3U/2 = \epsilon(n-1)(n-2)/2 < \epsilon n^2/2$.
- We fix such a u and take A to be all v such that $s(e_{uv}) = 1$, assigning the rest of V to B.

Counting 'bad' edges

- If e_{vw} is a negative edge in A, then the triangle on u, v, w is an unbalanced triangle containing u. There are at most $\epsilon n^2/2$ of those, and hence at most $\epsilon n^2/2$ negative edges in A.
- Similarly, if e_{vw} is a negative edge in B, then the triangle on u, v, w is unbalanced as it has three negative edges. So there are at most $\epsilon n^2/2$ negative edges in A.
- Further, if v is in one of A or B and w is in the other, with e_{vw} positive, then the triangle on u, v, w is unbalanced as it has a single negative edge. So there are at most $\epsilon n^2/2$ positive edges joining a vertex in A to one in B.

Case 1

- Let A contain a vertices, and B contain $b = n - a$.
- Suppose $a \geq (1-\delta)n$. Since $\epsilon < 1/8$, $\delta < \sqrt[3]{1/8} = 1/2$. So $a > n/2$ so $a \geq n/2 + 1$ (for n even; similar arguments will work for n odd).
- Therefore A has $\geq (n/2+1)(n/2)/2$ edges, i.e., at least $n^2/8$.
- At most $\epsilon n^2/2$ of those are negative, so the fraction of negative edges in A is at most

$$\frac{\epsilon n^2/2}{n^2/8} = 4\epsilon = 4\delta^3 = 4\delta^2\delta < \delta.$$

- So if $a \geq (1-\delta)n$, we can take $S = A$ satisfying case 1 of the theorem. Similarly, if $b \geq (1-\delta)n$ we can take $S = B$.

Case 2

- Otherwise $a < (1-\delta)n$ and $b < (1-\delta)n$. Since $n = a + b$, we also have $a > \delta n$ and $b > \delta n$.
- Expanding $n^2 = (a+b)^2$ and using $\delta < 1/2$, we have that

$$ab \geq (\delta n)(1 - \delta n) = \delta(1-\delta)n^2 \geq \delta n^2/2.$$

- But ab is precisely the number of edges with one end in A and the other in B; we know at most $\epsilon n^2/2$ of these are positive edges, so the fraction of positive edges is at most

$$\frac{\epsilon n^2/2}{\delta n^2/2} = \epsilon/\delta = \delta^2 < \delta.$$

- Meanwhile, there are $a(a-1)/2$ edges in A, and $a > \delta n$, so there are at least $\delta^2 n^2/2$ edges in A. At most $\epsilon n^2/2$ of these are negative edges, so the fraction of negative edges in A is at most

$$\frac{\epsilon n^2/2}{\delta^2 n^2/2} = \epsilon/\delta^2 = \delta.$$

  We have exactly the same property for B using b.

Example

Taking $\epsilon = 0.001$, we obtain $\delta = 0.1$ and thus:
If $G = (V, E)$ is a complete signed graph in which at least 99.9% of all triangles are balanced, then either:

1. There is a set S consisting of at least 90% of the vertices, in which at least 90% of all edges between elements of S are positive;
2. Or V can be partitioned into sets A, B such that
   - at least 90% of the edges in A are positive;
   - at least 90% of the edges in B are positive;
   - At least 90% of the edges with one endpoint in A and the other in B are negative.