Topics in Discrete Mathematics
Introduction to Graph Theory

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29/i/13

Yesterday...

We looked at basic definitions and properties, then considered **trails**:

- Can be open, or closed (a **tour**).
- Cannot re-use an edge.
- If it manages to use *every* edge, it's **Eulerian**.
- Existence of Eulerian tours / trails is easy to decide (restrictions on vertex degree parity).
- When they do exist, there's an efficient algorithm to construct them.

Yesterday...

We then moved on to **paths**:

- Can be open, or closed (a **cycle**).
- Cannot re-use an edge *or* a vertex.
- If it manages to use *every* vertex, it's **Hamiltonian**.
- Existence of Hamiltonian paths / cycles is (NP) hard! Statement of Dirac's theorem gave sufficient conditions, but we saw examples that proved they weren't necessary.

Theorem (Dirac's Theorem)

Let G be an n-vertex graph, where $n \geq 3$. Then if $d(v) \geq n/2$ for every vertex v of G, then G is Hamiltonian.

Proof of Dirac's Theorem

Proposition

If $G = (V, E)$ is Hamiltonian, then so is any graph obtained by adding edges to E (whilst keeping V fixed).

'Having a Hamiltonian cycle' is therefore an example of a *monotone increasing* graph property.

Proof of Dirac's Theorem

Suppose, for contradiction, that there are non-Hamiltonian, n-vertex graphs in which every vertex has degree at least $n/2$.

Let G be such a graph with the maximum possible number of edges:

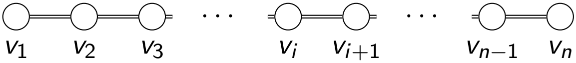
- G does not contain a Hamiltonian cycle;
- Adding *any* edge to G gives a graph containing a Hamiltonian cycle.

Proposition

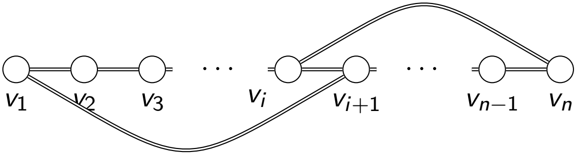
G contains a Hamiltonian path P.

Proof of Dirac's Theorem

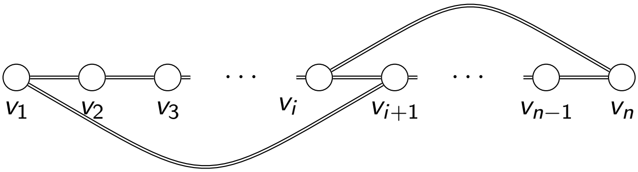
- Relabel the vertices of G so that

$P =$ 

- $\{v_1, v_n\} \notin E$, otherwise P is a Hamiltonian cycle in G.
- But v_1 has at least $n/2$ neighbours, as does v_n, and these must all be drawn from the set $\{v_2, \ldots v_{n-1}\}$.
- So there exists $v_i \in \{v_2, \ldots, v_{n-1}\}$ that neighbours v_n, with v_{i+1} neighbouring v_1:



Proof of Dirac's Theorem



In other words, the sequence

$$v_1, v_2, \ldots, v_i, v_n, v_{n-1}, \ldots, v_{i+1}, v_1$$

is a Hamiltonian cycle in G, contradicting the assumption it was non-Hamiltonian.

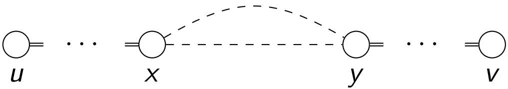
Definition

A connected graph with no cycles is called a **tree**; if G has no cycles, but is not necessarily connected, then it is a **forest**.

Lemma

G is a tree if and only if $G = (V, E)$ is a graph such that for all $u, v \in V$ there is a unique path from u to v.

$\Rightarrow$ G is connected, so there is at least one path P_1 from u to v. If there is a second, P_2, let x be the first vertex where the paths diverge, and y the next vertex they share.



Then there are two paths from x to y with no common edges, so there is a cycle from x to y.
$\Leftarrow$ G is connected. But G cannot contain a cycle, else picking any two vertices on that cycle we have distinct paths between them by following the cycle in opposite directions.

Lemma

$G = (V, E)$ is a tree if and only if G is connected, but removing any edge of G makes it disconnected.

$\Rightarrow$ By definition G is connected. Let $e_{uv} = \{u, v\} \in E$. Then e_{uv} is a path from u to v, and it is the only one. So deleting e_{uv} disconnects G since there is no path from u to v.
$\Leftarrow$ We have connectivity by assumption. If there were a cycle in G starting at u, with v the next vertex in one direction, then we could delete e_{uv} without disconnecting G, since we could follow the cycle in the opposite direction from u to v.

Lemma

$G = (V, E)$ is a tree if and only if G has no cycles, but adding any edge to G creates a cycle.

$\Rightarrow$ G has no cycles; we must show that adding any edge $\{u, v\} \notin E$ creates one. But there is a unique path from v to u, so adding e_{uv} would create a cycle at v.
$\Leftarrow$ By assumption G is cycle-free, so we need only show connectivity. But it cannot be disconnected, for if it were, picking vertices u, v from different connected components would allow us to add the edge $\{u, v\}$ without creating a cycle.

Lemma

$G = (V, E)$ is a tree iff G is connected and $|E| = |V| - 1$.

$\Rightarrow$

- Suppose we have a k-vertex tree. If $k = 1, 2$ or 3, then clearly $|E| = n - 1$. Otherwise we proceed by strong induction.
- Hypothesis: any k-vertex tree with $k \leq n$ has $k - 1$ edges.
- Let G be an $(n + 1)$-vertex tree. Deleting any edge from G disconnects it into $l-$ and m-vertex graphs G_l, G_m with $1 \leq l, m \leq n$.
- G_l, G_m are trees, and hence have $l - 1$ and $m - 1$ edges respectively.
- So G has $1 + (l - 1) + (m - 1) = l + m - 1 = n$ edges.

Lemma

$G = (V, E)$ is a tree iff G is connected and $|E| = |V| - 1$.

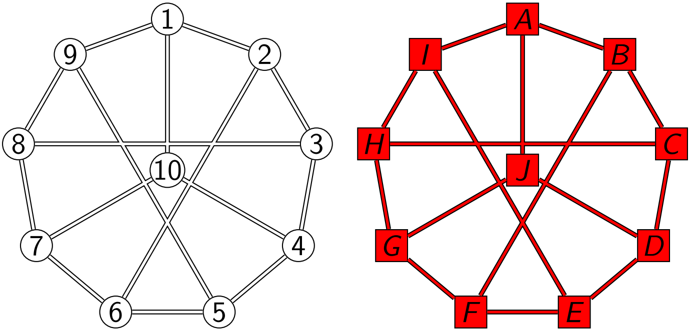
$\Leftarrow$ If G is an n-vertex graph with $n - 1$ edges, then deleting any edge from G gives an n-vertex graph with $n - 2$ edges, which cannot be connected. So G is a tree.

Recognising Trees

The following are therefore all equivalent for a graph $G = (V, E)$:

- G is connected and contains no cycles;
- G For any $u, v \in V$, there is a unique path from u to v;
- G is connected, but removing any edge makes it disconnected;
- G has no cycles, but adding any edge creates a cycle;
- G is connected and $|E| = |V| - 1$.

Spot the difference!



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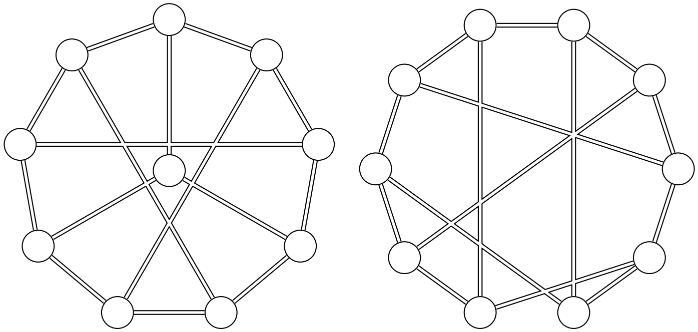

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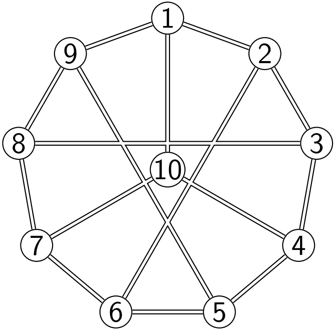

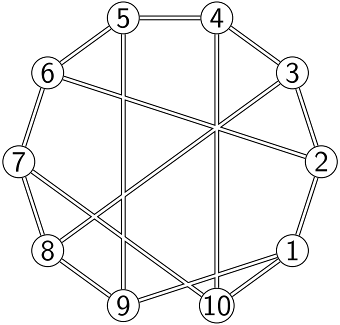

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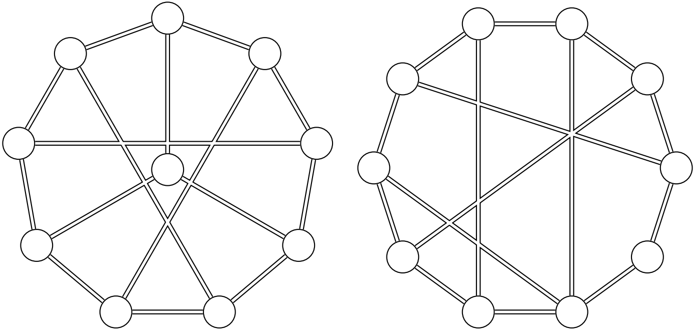

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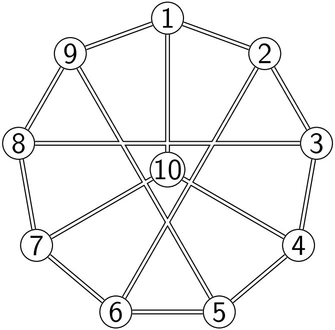

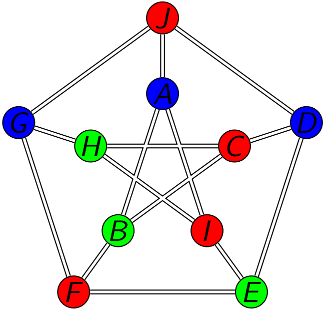

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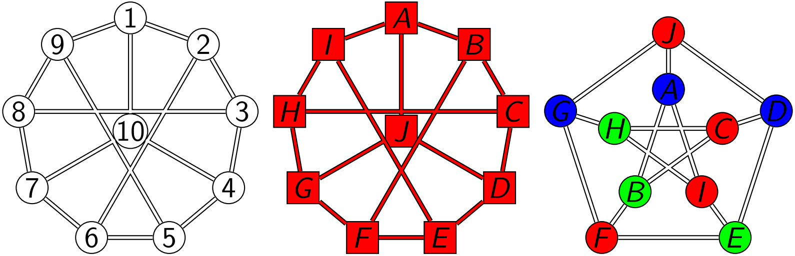
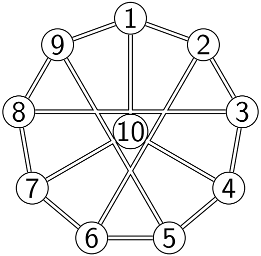
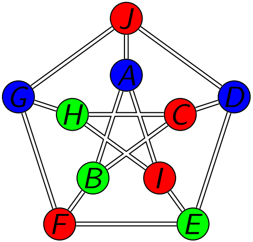


Definition

Let $G_1 = G(V_1, E_1)$ and $G_2 = G(V_2, E_2)$ be two graphs. We say that G_1 and G_2 are **isomorphic** if there exists a bijection $f : V_1 \to V_2$ such that $\{v_1, v_2\} \in E_1 \Leftrightarrow \{f(v_1), f(v_2)\} \in E_2$. Such an f is called a **graph isomorphism**.





$V_1 = \{1, \ldots, 10\}$ $V_2 = \{A, \ldots, J\}$

$$\begin{aligned} E_1 &= \{\{1,2\},\{1,9\},\{1,10\},\{2,3\},\{2,6\},\{3,4\},\{3,8\},\{4,5\},\{4,10\}, \\ &\quad \{5,6\},\{5,9\},\{6,7\},\{7,8\},\{7,10\},\{8,9\}\} \\ E_2 &= \{\{A,B\},\{A,I\},\{A,J\},\{B,C\},\{B,F\},\{C,D\},\{C,H\},\{D,E\}, \\ &\quad \{D,J\},\{E,F\},\{E,I\},\{F,G\},\{G,H\},\{G,J\},\{H,I\}\}. \end{aligned}$$

The map f that sends $1 \leftrightarrow A, \ldots, 10 \leftrightarrow J$ is clearly a bijection, and satisfies $\{i,j\} \in V_1 \Leftrightarrow \{f(i), f(j)\} \in V_2$.

Definition

Let $G = (V, E)$ be an n-vertex graph. The **adjacency matrix** of G is the n-by-n matrix **A** given by

$$A_{ij} = \begin{cases} 1 & \text{If } \{v_i, v_j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

Corollary

For the simple undirected graphs we are considering, the adjacency matrix is a symmetric $\{0, 1\}$*-matrix with zeroes on the diagonal.*

Problem!

- The ith row of the matrix describes the neighbours of vertex v_i.
- The adjacency matrix will vary depending on the ordering assigned to the vertices!
- So which is the ‘correct’ adjacency matrix of a graph?
- Fortunately, any rival adjacency matrix produced in this way will be related to the original in a precise manner, so it doesn’t matter:

Proposition

For each isomorphism class of graphs, the adjacency matrix is unique up to permutation of the rows and columns (equivalently, if $\mathbf{A_1}, \mathbf{A_2}$ are adjacency matrices for G, then $\mathbf{A_1} = \mathbf{P}^{-1}\mathbf{A_2}\mathbf{P}$ for some permutation matrix $\mathbf{P}$, corresponding to an alternative ordering of the vertices of G).

- For each isomorphism class of graphs, we have an equivalence class of matrices (under the operation of conjugation by a permutation matrix).
- *Any* matrix from that class serves as a representative of the isomorphism class of graphs.
- Two graphs are isomorphic if and only if their adjacency matrices are equivalent.