Topics in Discrete Mathematics
Introduction to Graph Theory

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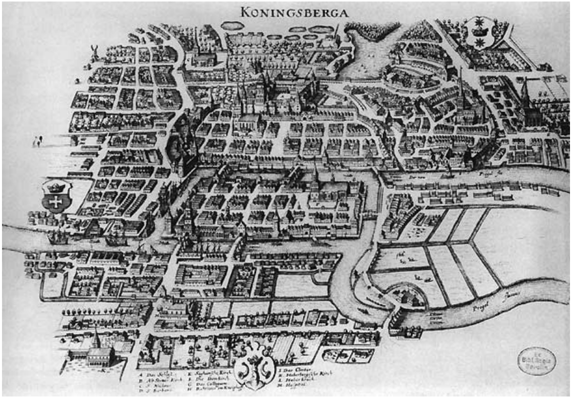
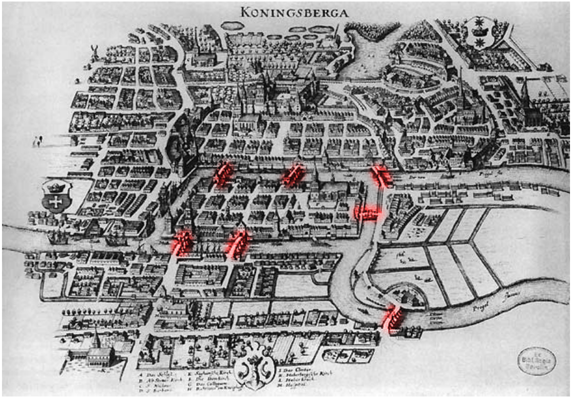
28/i/13

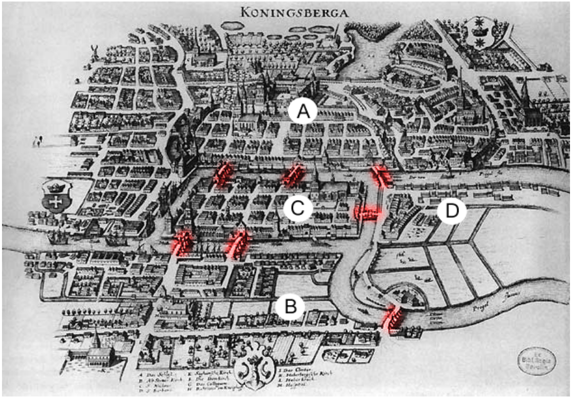
The Seven Bridges of Königsberg



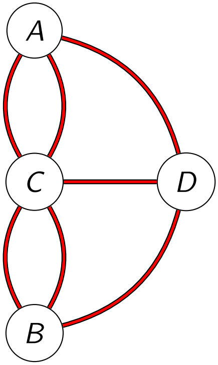
The Seven Bridges of Königsberg



The Seven Bridges of Königsberg

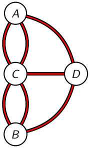


The Seven Edges of Königsberg



Solving the puzzle

- A solution would be a **trail** of vertices, such that each step between **adjacent** vertices is via a previously unused edge, and every edge is used somewhere along the trail.
- To **walk** between two vertices such as A and B via another such as C, we require a pair of edges to be **incident** at C- one to get there from A, another to continue to B.
- So, apart from perhaps the start and end vertices, each must have an even number of edges incident at it.
- Sadly for the people of Königsberg, all four regions are home to an odd number of bridges.



Definition

A **(simple) graph** G is given by a finite set V of **vertices** and a set E whose elements are subsets of two (distinct) elements of V, the **edges** of G.
If $e = \{v_i, v_j\} \in E$ then we describe the vertices v_i, v_j as being **adjacent**, and say that the edge e is **incident** at vertices v_i and v_j, its **endpoints**.
We denote G by $G(V, E)$.

Definition

The **degree** $d(v)$ of a vertex v is the number of edges that are incident at v or, equivalently, the number of vertices it is adjacent to.

Lemma (Handshaking Lemma)

For any graph $G = G(V, E)$,

$$\sum_{v \in V} d(v) = 2 \times \#E.$$

Corollary

The total degree of a graph is even.

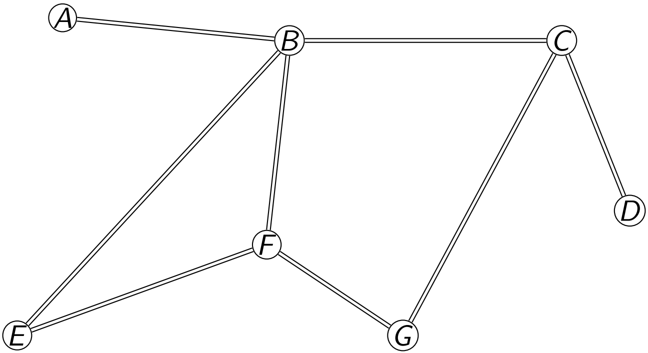
Corollary

For any graph, the number of vertices with odd degree is even.

Definition

A **walk** is a sequence $v_1, e_1, v_2, e_2, \ldots v_n$ of vertices and edges such that $e_i = \{v_i, v_{i+1}\} \in E$ for all i.
The length of the walk is $n-1$, the number of edges used.
If the end vertex v_n is the start vertex v_1, then the walk is **closed**; otherwise, it is **open**.

"I travel not to go anywhere, but to go. I travel for travel's sake. The great affair is to move" - Robert Louis Stevenson.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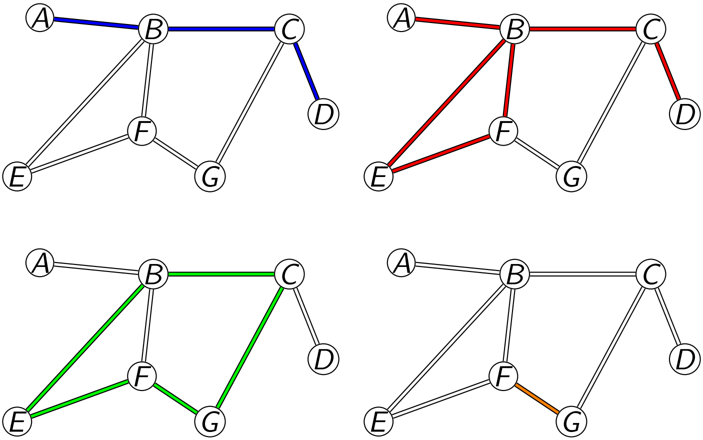
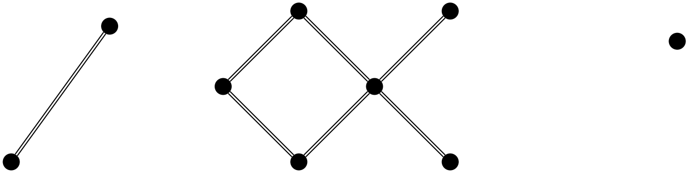
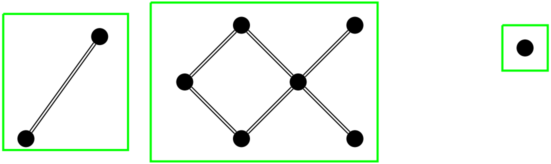


A walk of length 3 from A to D; a walk of length 6 from A to D; a closed walk of length 5 starting at B; and a closed walk $F - G - F$ of length 2.



Definition

A graph $G(V, E)$ is **connected** if for every $v, w \in V$ there exists a walk with v as the start vertex and w the end vertex. A **disconnected** graph consists of a number of **connected components**, whereby any two vertices in the same component have a walk between them.



A disconnected graph with three connected components.

Definition

A **trail** is a walk that does not reuse any edge (So $|\{e_1, \ldots, e_n\}| = n$). If it is closed, then it is called a **tour**.
A trail that uses every edge of G (so each edge appears once and only once in the trail) is described as **Eulerian**.

Theorem

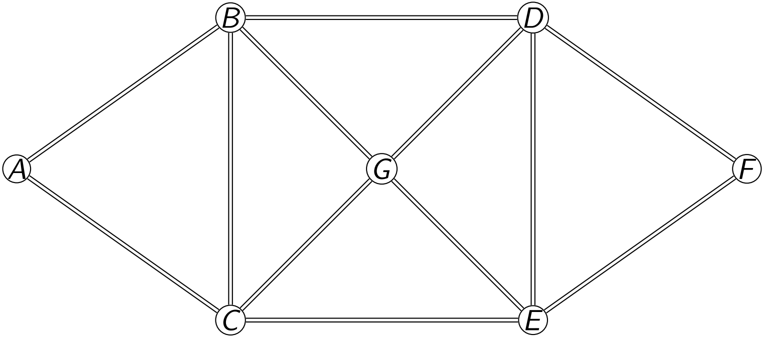
A connected graph G has an Eulerian **tour** *if and only if all vertices have even degree.*

We first prove that if there is an Eulerian tour, then every vertex has even degree.

- Let v be a vertex of interest, and start the tour at any $w \neq v$.
- Each time a tour visits v it must reach it via an edge that hasn't been used before, and then leave it by another edge that hasn't been used before.
- To be an Euler tour, every edge incident at v is used precisely once, during some visit to v.
- If the total number of visits to v is x, then $2x$ edges incident to v are used during those visits, and each edge incident to v is one of those edges.
- So $d(v) = 2x$ which is obviously even.

Now suppose every vertex of G has even degree. We will prove by construction that G has an Euler tour T.

1. Pick any vertex v and construct a tour T that starts and ends at v.
2. If T uses every edge in G, it is Eulerian and we are done.
3. If not, there are edges of G not in T: consider the subgraph $G' = G \backslash T$. Every vertex of G' has even degree, and there is at least one vertex w in both T and G'.
4. Construct a tour T' in G' starting and ending at w, then **splice** it into T by starting at v, following T until we reach w, then following T' until we return to w, at which point we resume T until arriving back at v.
5. This gives a strictly larger tour $T = T \cup T'$, and strictly smaller $G' = G \backslash (T \cup T')$.
6. Then T is either Eulerian or we may return to step 3 and iterate; eventually all edges must be used, and so T must be Eulerian.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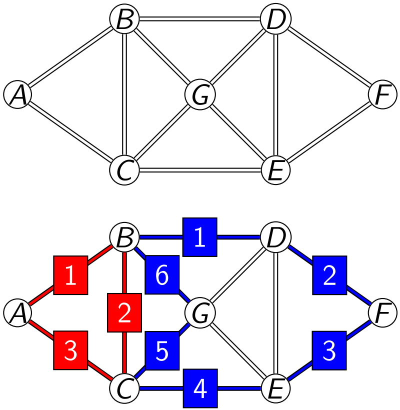
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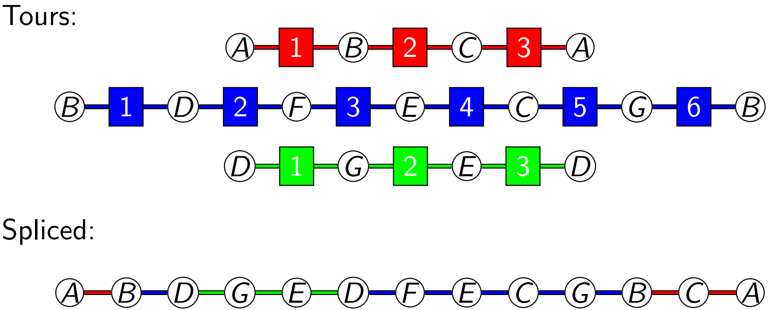


Tours:
A 1 B 2 C 3 A
B 1 D 2 F 3 E 4 C 5 G 6 B
D 1 G 2 E 3 D
Spliced:
A B D G E D F E C G B C A

Corollary

A graph G has an Eulerian **trail** *if it has either zero or two vertices of odd degree.*

- If G has zero vertices of odd degree, it has an Euler tour.
- It cannot have exactly one vertex of odd degree, by handshaking.

- If there are exactly two vertices v_1, v_2 of odd degree, then we may introduce a dummy edge $\{v_1, v_2\}$ to obtain a graph with all vertices of even degree: so it has an Euler tour.
- By starting that tour at v_1 and following it to v_2 along the dummy edge, we can then proceed to visit every edge before returning to v_1. Thus there is an Euler trail in the original graph G that starts at v_2 and ends at v_1.
- If there were an Eulerian trail in a graph with at least 3 vertices of odd degree, the same trick would give us an Eulerian tour in a graph with at least 1 vertex of odd degree, which cannot exist.

Recap

- A **walk** is a sequence $v_1, e_1, v_2, e_2, \ldots v_n$ of vertices and edges such that $e_i = \{v_i, v_{i+1}\} \in E$ for all i.
- In a **closed walk**, $v_1 = v_n$.
- A **trail** is a walk that does not reuse any edge.
- A **tour** is a closed walk that does not reuse any edge.
- An **Eulerian** trail/tour uses *every* edge of a graph.
- There are simple necessary and sufficient conditions for the existence of Eulerian trails/tours; and when they exist, there is an efficient algorithm for constructing them.

What if we want to visit each of the *vertices* instead?

Definition

A **path** is a walk that never revisits a vertex- which prevents it revisiting any edge, so it is also a trail.
A **cycle** is a closed path: that is, it starts and ends at the same vertex, but otherwise has no repeated vertices *or* edges.

Definition

A path or cycle that visits every vertex of G is described as **Hamiltonian**. G itself is called **Hamiltonian** if it contains a Hamiltonian cycle.
If there is a Hamiltonian path between *every* pair of vertices of a graph G, it is called **Hamiltonian-connected**.

Warning!

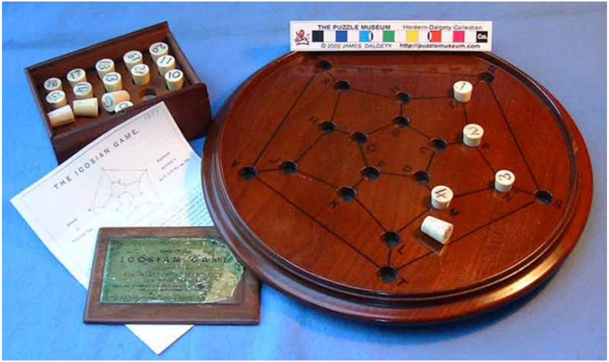
The **Hamiltonian path problem** and **Hamiltonian cycle problem** may look similar to the Euler trail / tour problems, but are very much harder. Specifically, they are *NP-complete*.

- It's easy to verify if a given listing of the vertices is a legal Hamiltonian path / cycle;
- But given G, there is no known polynomial-time algorithm to even decide if G contains a Hamiltonian path / cycle - never mind finding one!
- This remains true even if we put some strong restrictions on G.
- But under some conditions, there is *always* a Hamiltonian cycle:

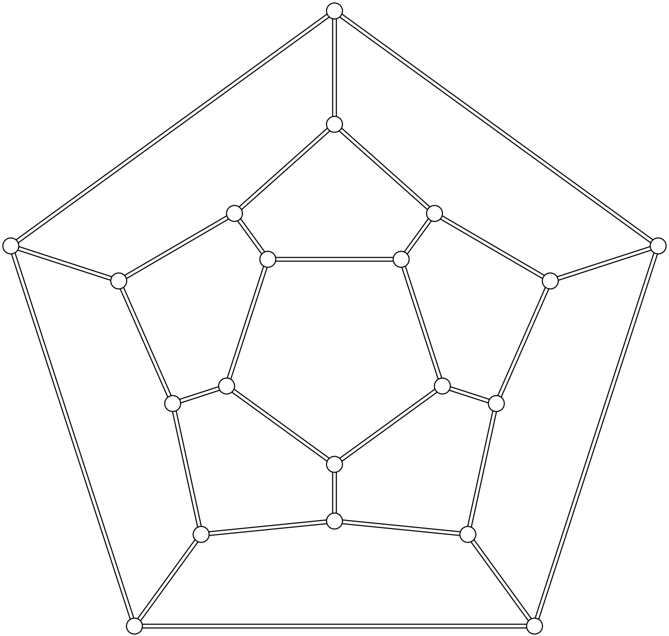
Theorem (Dirac's Theorem)

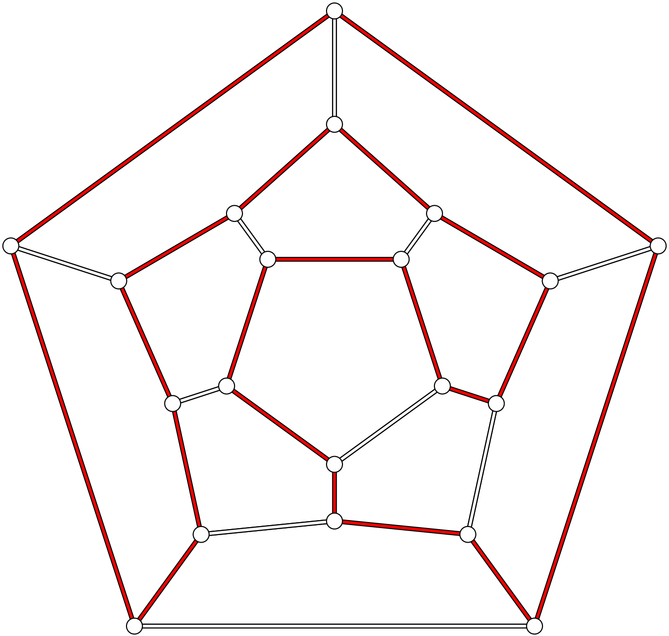
Let G be an n-vertex graph, where $n \geq 3$. Then if $d(v) \geq n/2$ for every vertex v of G, then G is Hamiltonian.

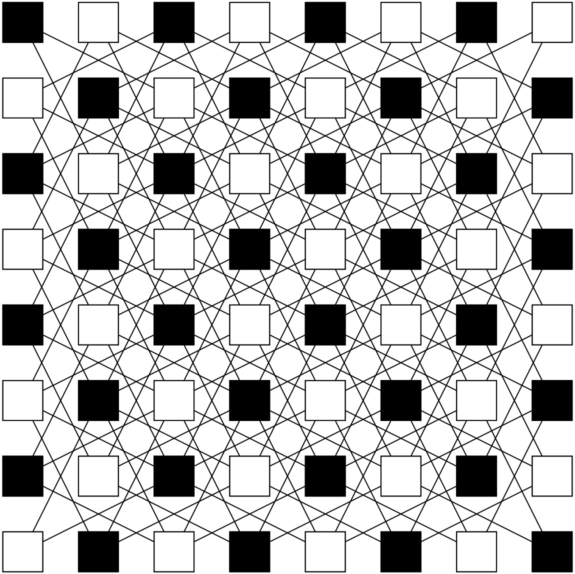
Dirac's theorem gives sufficient, but not necessary, conditions for being Hamiltonian.



William Rowan Hamilton's *Icosian Game*

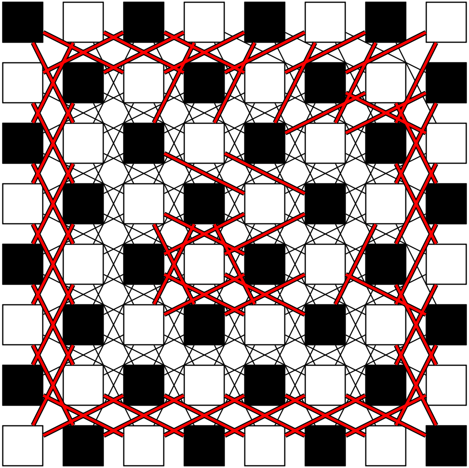






The graph of possible moves of a knight on an 8×8 chessboard.

It is known that there are 13,267,364,410,532 Hamiltonian cycles on the 8×8 Knight's graph (undirected, but allowing rotation and reflection).



However, the number of Hamiltonian paths on this graph is still an open question!