Cyclotomic Matrices and Graphs

Graeme Taylor

HIMR

31/i/2011

Mahler Measure

Let $P(z) = z^d + \cdots + a_d = \prod_{i=1}^{d}(z - \alpha_i) \in \mathbb{C}[z]$ be monic, non-constant.

Mahler Measure

Let $P(z) = z^d + \cdots + a_d = \prod_{i=1}^{d}(z - \alpha_i) \in \mathbb{C}[z]$ be monic, non-constant.

Definition

The *Mahler Measure* $M(P)$ is given by

$$M(P) := \exp\left(\int_0^1 \log|P(e^{2\pi i t})|dt\right)$$

Mahler Measure

Let $P(z) = z^d + \cdots + a_d = \prod_{i=1}^{d}(z - \alpha_i) \in \mathbb{C}[z]$ be monic, non-constant.

Definition

The *Mahler Measure* $M(P)$ is given by

$$M(P) := \exp\left(\int_0^1 \log|P(e^{2\pi it})|dt\right)$$

Equivalently,

$$M(P) = \prod_{i=1}^{d} \max(1, |\alpha_i|)$$

Mahler Measure

- Clearly, $M(P) \geq 1$ for all P.

Mahler Measure

- Clearly, $M(P) \geq 1$ for all P.
- $M(P) = 1 \Rightarrow$ all roots of P lie in the closed unit disc.

Mahler Measure

- Clearly, $M(P) \geq 1$ for all P.
- $M(P) = 1 \Rightarrow$ all roots of P lie in the closed unit disc.
- $\forall \lambda \geq 1$, $\exists P$ s.t. $M(P) = \lambda$.

Mahler Measure

Let $P(z) \in \mathbb{Z}[z]$. Then:

Mahler Measure

Let $P(z) \in \mathbb{Z}[z]$. Then:

- $M(P) \geq 1$ for all P.

Mahler Measure

Let $P(z) \in \mathbb{Z}[z]$. Then:

- $M(P) \geq 1$ for all P.
- $M(P) = 1 \Leftarrow P$ cyclotomic.

Mahler Measure

Let $P(z) \in \mathbb{Z}[z]$. Then:

- $M(P) \geq 1$ for all P.
- $M(P) = 1 \Leftrightarrow P$ cyclotomic (ish).

Mahler Measure

Let $P(z) \in \mathbb{Z}[z]$. Then:

- $M(P) \geq 1$ for all P.
- $M(P) = 1 \Leftrightarrow$ all nonzero roots of P are roots of unity.

Lehmer's Conjecture

- Lehmer's Problem: For a polynomial with $M(P) > 1$, can $M(P)$ be arbitrarily close to 1?

Lehmer's Conjecture

- Lehmer's Problem: For a polynomial with $M(P) > 1$, can $M(P)$ be arbitrarily close to 1?
- If not, then there exists some $\lambda > 1$ such that

$$M(P) > 1 \Rightarrow M(P) \geq \lambda.$$

Lehmer's Conjecture

The smallest known Mahler measure greater than 1 for a monic polynomial from $\mathbb{Z}[z]$ is

$$\lambda_0 = 1.176280818\ldots$$

Lehmer's Conjecture

The smallest known Mahler measure greater than 1 for a monic polynomial from $\mathbb{Z}[z]$ is

$$\lambda_0 = 1.176280818\ldots$$

which is the larger real root of the *Lehmer polynomial*

$$z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

Lehmer's Conjecture

The smallest known Mahler measure greater than 1 for a monic polynomial from $\mathbb{Z}[z]$ is

$$\lambda_0 = 1.176280818\ldots$$

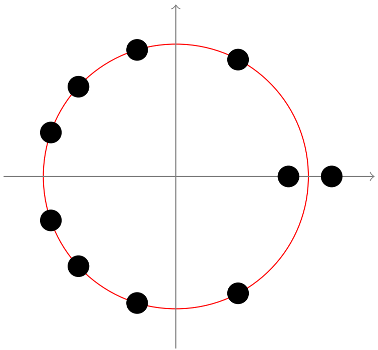
which is the larger real root of the *Lehmer polynomial*

$$z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

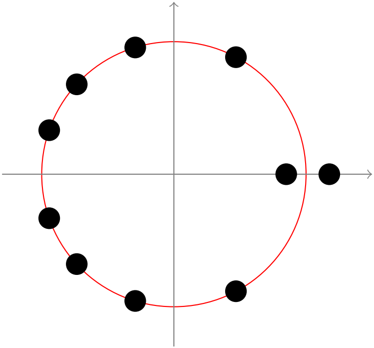
Current conjecture

$$M(P) > 1 \Rightarrow M(P) \geq \lambda_0$$

From cyclotomic to noncyclotomic?



From cyclotomic to noncyclotomic?



Likely candidates for small Mahler measure are polynomials that are 'almost cyclotomic'.

Associated Polynomials

Associated Polynomials

- If A is an $n \times n$ integer symmetric matrix, then its *associated polynomial* is $R_A(z) := z^n \chi_A(z + 1/z)$

Associated Polynomials

- If A is an $n \times n$ integer symmetric matrix, then its *associated polynomial* is $R_A(z) := z^n \chi_A(z + 1/z)$
- If A has all eigenvalues in $[-2, 2]$, then $M(R_A) = 1$
  (We describe A as a *cyclotomic matrix*.)

From cyclotomic to noncyclotomic

Theorem (Cauchy Interlacing Theorem)

Let A be a Hermitian $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

From cyclotomic to noncyclotomic

Theorem (Cauchy Interlacing Theorem)

Let A be a Hermitian $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Let B be obtained from A by deleting row i and column i from A.

From cyclotomic to noncyclotomic

Theorem (Cauchy Interlacing Theorem)

Let A be a Hermitian $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Let B be obtained from A by deleting row i and column i from A. Then the eigenvalues $\mu_1 \leq \cdots \leq \mu_{n-1}$ of B interlace with those of A: that is,

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n$$

From cyclotomic to noncyclotomic

We can run this process in reverse. Let B be a cyclotomic matrix, so its eigenvalues satisfy

$$-2 \leq \mu_1 \leq \cdots \leq \mu_{n-1} \leq 2$$

From cyclotomic to noncyclotomic

We can run this process in reverse. Let B be a cyclotomic matrix, so its eigenvalues satisfy

$$-2 \leq \mu_1 \leq \cdots \leq \mu_{n-1} \leq 2$$

Then if we 'grow' a matrix A from B by adding an extra row and column, we have by interlacing

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n$$

From cyclotomic to noncyclotomic

We can run this process in reverse. Let B be a cyclotomic matrix, so its eigenvalues satisfy

$$-2 \leq \mu_1 \leq \cdots \leq \mu_{n-1} \leq 2$$

Then if we 'grow' a matrix A from B by adding an extra row and column, we have by interlacing

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n$$

So

$$\lambda_2, \ldots, \lambda_{n-1} \in [\mu_1, \mu_{n-1}] \subseteq [-2, 2]$$

From cyclotomic to noncyclotomic

We can run this process in reverse. Let B be a cyclotomic matrix, so its eigenvalues satisfy

$$-2 \leq \mu_1 \leq \cdots \leq \mu_{n-1} \leq 2$$

Then if we 'grow' a matrix A from B by adding an extra row and column, we have by interlacing

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n$$

So

$$\lambda_2, \ldots, \lambda_{n-1} \in [\mu_1, \mu_{n-1}] \subseteq [-2, 2]$$

At worst,

$$\lambda_1, \lambda_n \notin [-2, 2]$$

From cyclotomic to noncyclotomic

$$\begin{pmatrix} 1 & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

1

From cyclotomic to noncyclotomic

$$\begin{pmatrix} 1 & 1 & & \\ 1 & \text{-}1 & & \\ & & & \\ & & & \end{pmatrix}$$

$$-\sqrt{2} \leq \quad 1 \quad \leq \sqrt{2}$$

From cyclotomic to noncyclotomic

$$\begin{pmatrix} 1 & 1 & 0 & & \\ 1 & -1 & 1 & & \\ 0 & 1 & 0 & & \\ & & & & \\ & & & & \end{pmatrix}$$

$$-1.88 \le -\sqrt{2} \le 0.35 \le \sqrt{2} \le 1.53$$

From cyclotomic to noncyclotomic

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$-2 \leq -1.88 \leq -0.62 \leq 0.35 \leq 1 \leq 1.53 \leq 1.62$$

From cyclotomic to noncyclotomic

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$-2.03 \leq -2 \leq -0.91 \leq -0.62 \leq 0.58 \leq 1 \leq 1.47 \leq 1.62 \leq 1.89$$

Entries

Proposition

The only cyclotomic 1×1 *matrices are*

$$(0), (1), (-1), (2), (-2)$$

Entries

Proposition

The only cyclotomic 1×1 *matrices are*

$$(0), (1), (-1), (2), (-2)$$

Corollary

By interlacing, the entries of an integer cyclotomic matrix must be elements of $\{0, 1, -1, 2, -2\}$.

Indecomposability

If M decomposes as a block-diagonal matrix, then its eigenvalues are those of the blocks.

Indecomposability

If M decomposes as a block-diagonal matrix, then its eigenvalues are those of the blocks. Thus a cyclotomic matrix decomposes into one or more indecomp. cyclotomic matrices.

Indecomposability

If M decomposes as a block-diagonal matrix, then its eigenvalues are those of the blocks. Thus a cyclotomic matrix decomposes into one or more indecomp. cyclotomic matrices.

Lemma

Apart from the matrices

$$(2), (-2), \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

any indecomp. cyclotomic matrix has all entries from $\{0, 1, -1\}$.

Maximality

- If A is cyclotomic, so is any B obtained by deleting some set of rows and corresponding columns of A: B is described as being *contained in* A.

Maximality

- If A is cyclotomic, so is any B obtained by deleting some set of rows and corresponding columns of A: B is described as being *contained in* A.
- If M is an indecomp. cyclotomic matrix that is not contained in any strictly larger indecomp. cyclotomic matrix, then M is described as being *maximal*.

Maximality

- If A is cyclotomic, so is any B obtained by deleting some set of rows and corresponding columns of A: B is described as being *contained in* A.
- If M is an indecomp. cyclotomic matrix that is not contained in any strictly larger indecomp. cyclotomic matrix, then M is described as being *maximal*.
- Theorem (McKee, Smyth): Any non-maximal indecomp. cyclotomic matrix is contained in a maximal one.

Equivalence

Let $O_n(\mathbb{Z})$ be the orthogonal group of $n \times n$ signed permutation matrices, generated by permutation matrices and matrices of the form

$$diag(1, 1, \ldots, 1, -1, 1, \ldots, 1)$$

Equivalence

Let $O_n(\mathbb{Z})$ be the orthogonal group of $n \times n$ signed permutation matrices, generated by permutation matrices and matrices of the form

$$diag(1, 1, \ldots, 1, -1, 1, \ldots, 1)$$

- If A is cyclotomic and $X \in O_n(\mathbb{Z})$, then $M' = XAX^{-1}$ is cyclotomic since it has the same eigenvalues. We describe A and A' as *strongly equivalent*.

Equivalence

Let $O_n(\mathbb{Z})$ be the orthogonal group of $n \times n$ signed permutation matrices, generated by permutation matrices and matrices of the form

$$diag(1, 1, \ldots, 1, -1, 1, \ldots, 1)$$

- If A is cyclotomic and $X \in O_n(\mathbb{Z})$, then $M' = XAX^{-1}$ is cyclotomic since it has the same eigenvalues. We describe A and A' as *strongly equivalent*.
- A matrix A' is then described as *equivalent* to A if it is strongly equivalent to either A or $-A$.

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

Charged Signed Graph

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- $n \times n$ Matrix A

Charged Signed Graph

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- $n \times n$ Matrix A

Charged Signed Graph

- n-vertex Graph G

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- $n \times n$ Matrix A
- $A_{ij} = 1$

Charged Signed Graph

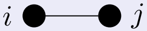
- n-vertex Graph G

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- $n \times n$ Matrix A
- $A_{ij} = 1$

Charged Signed Graph

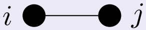
- n-vertex Graph G
- Edge 

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- $n \times n$ Matrix A
- $A_{ij} = 1$
- $A_{ij} = -1$

Charged Signed Graph

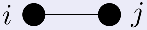
- n-vertex Graph G
- Edge 

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- $n \times n$ Matrix A
- $A_{ij} = 1$
- $A_{ij} = -1$

Charged Signed Graph

- n-vertex Graph G
- Edge 

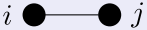
- Negative edge 

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- $n \times n$ Matrix A
- $A_{ij} = 1$
- $A_{ij} = -1$
- $A_{ii} = 0$

Charged Signed Graph

- n-vertex Graph G
- Edge 

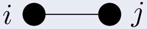
- Negative edge 

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- $n \times n$ Matrix A
- $A_{ij} = 1$
- $A_{ij} = -1$
- $A_{ii} = 0$

Charged Signed Graph

- n-vertex Graph G
- Edge 

- Negative edge 

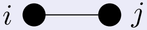
- Neutral Vertex 

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- $n \times n$ Matrix A
- $A_{ij} = 1$
- $A_{ij} = -1$
- $A_{ii} = 0$
- $A_{ii} = 1$

Charged Signed Graph

- n-vertex Graph G
- Edge 

- Negative edge 

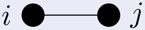
- Neutral Vertex 

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- $n \times n$ Matrix A
- $A_{ij} = 1$
- $A_{ij} = -1$
- $A_{ii} = 0$
- $A_{ii} = 1$

Charged Signed Graph

- n-vertex Graph G
- Edge 

- Negative edge 

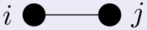
- Neutral Vertex ●
- Positive Vertex $\oplus$

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- $n \times n$ Matrix A
- $A_{ij} = 1$
- $A_{ij} = -1$
- $A_{ii} = 0$
- $A_{ii} = 1$
- $A_{ii} = -1$

Charged Signed Graph

- n-vertex Graph G
- Edge 

- Negative edge 

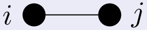
- Neutral Vertex ●
- Positive Vertex ⊕

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- $n \times n$ Matrix A
- $A_{ij} = 1$
- $A_{ij} = -1$
- $A_{ii} = 0$
- $A_{ii} = 1$
- $A_{ii} = -1$

Charged Signed Graph

- n-vertex Graph G
- Edge 

- Negative edge 

- Neutral Vertex ●
- Positive Vertex ⊕
- Negative Vertex ⊖

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

Charged Signed Graph

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- A indecomposable.

Charged Signed Graph

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- A indecomposable.

Charged Signed Graph

- G connected.

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- A indecomposable.
- A' contained in A.

Charged Signed Graph

- G connected.

Charged Signed Graphs

Symmetric $\{0,1,-1\}$-matrix

- A indecomposable.
- A' contained in A.

Charged Signed Graph

- G connected.
- G' induced subgraph of G.

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- A indecomposable.
- A' contained in A.
- A_1 permutation of A_2.

Charged Signed Graph

- G connected.
- G' induced subgraph of G.

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- A indecomposable.
- A' contained in A.
- A_1 permutation of A_2.

Charged Signed Graph

- G connected.
- G' induced subgraph of G.
- G_1 vertex relabelling of G_2.

Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- A indecomposable.
- A' contained in A.
- A_1 permutation of A_2.
- Conjugation by kth diagonal matrix.

Charged Signed Graph

- G connected.
- G' induced subgraph of G.
- G_1 vertex relabelling of G_2.

Charged Signed Graphs

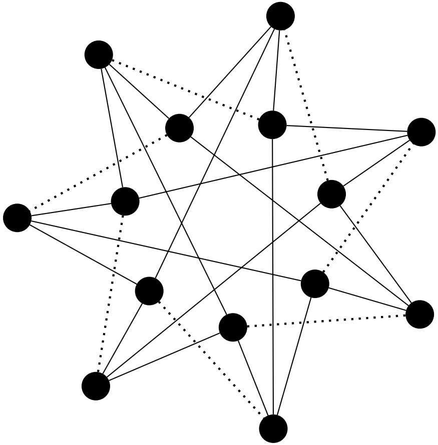
Symmetric $\{0, 1, -1\}$-matrix

- A indecomposable.
- A' contained in A.
- A_1 permutation of A_2.
- Conjugation by kth diagonal matrix.

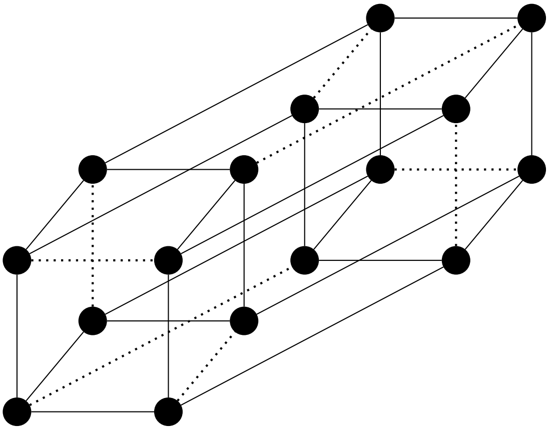
Charged Signed Graph

- G connected.
- G' induced subgraph of G.
- G_1 vertex relabelling of G_2.
- Switching of signs of all edges at vertex k.

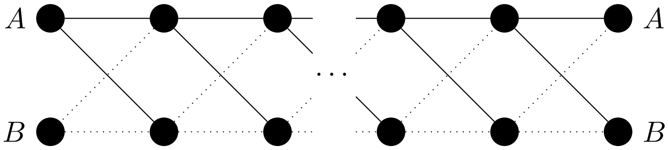
The maximal connected cyclotomic signed graph S_{14}.



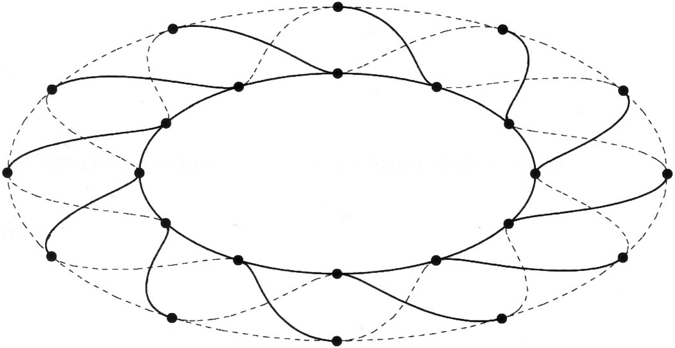
The maximal connected cyclotomic signed graph S_{16}.



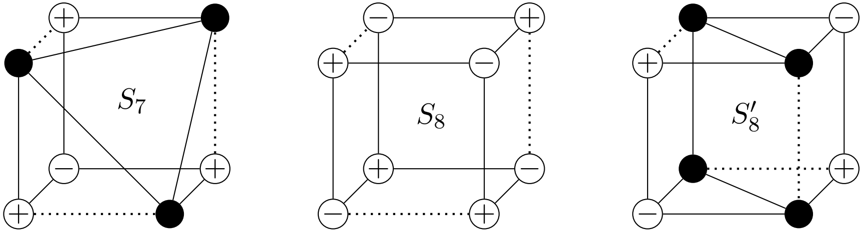
The maximal connected cyclotomic signed graphs T_{2k}.



Example: T_{24}

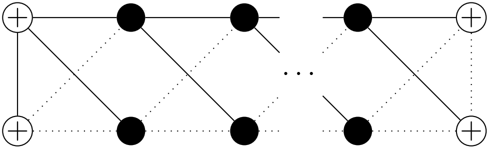


The maximal connected cyclotomic charged signed graphs S_7, S_8, S_8'.

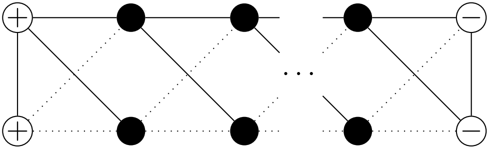


The maximal connected cyclotomic charged signed graphs C_{2k}^{++} and C_{2k}^{+-}.

$C_{2k}^{++} :=$



$C_{2k}^{+-} :=$



From cyclotomic to noncyclotomic

Theorem (McKee,Smyth)

If A is a noncyclotomic integer symmetric matrix then

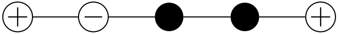
$$M(R_A(z)) \geq \lambda_0$$

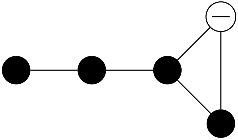
From cyclotomic to noncyclotomic

Theorem (McKee,Smyth)

If A is a noncyclotomic integer symmetric matrix then

$$M(R_A(z)) \geq \lambda_0$$



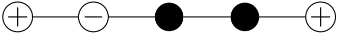


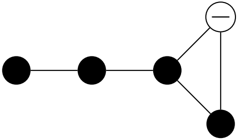
From cyclotomic to noncyclotomic

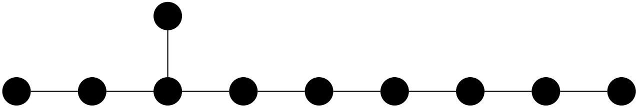
Theorem (McKee,Smyth)

If A is a noncyclotomic integer symmetric matrix then

$$M(R_A(z)) \geq \lambda_0$$







Nonreciprocal Polymonials

Reciprocality

For any ISM A, $R_A(z)$ is a reciprocal polynomial.

Nonreciprocal Polymonials

Reciprocality

For any ISM A, $R_A(z)$ is a reciprocal polynomial.

Theorem (Smyth)

Let α be a nonreciprocal algebraic integer with minimal polynomial P_α. Then

$$M(\alpha) := M(P_\alpha) \geq M(z^3 - z - 1) = \theta_0 = 1.3247\ldots$$

Nonreciprocal Polymonials

Reciprocality

For any ISM A, $R_A(z)$ is a reciprocal polynomial.

Theorem (Smyth)

Let α be a nonreciprocal algebraic integer with minimal polynomial P_α. Then

$$M(\alpha) := M(P_\alpha) \geq M(z^3 - z - 1) = \theta_0 = 1.3247\ldots$$

Corollary

Lehmer's Conjecture holds for nonreciprocal polynomials.

Characteristic Polynomials

Question

Given a monic reciprocal $P \in \mathbb{Z}[z]$, does there exist an ISM A such that

$$P = R_A(z)?$$

Characteristic Polynomials

Question

Given a monic reciprocal $P \in \mathbb{Z}[z]$, does there exist an ISM A such that

$$P = R_A(z)?$$

Lemma

There do not exist $a, b, c \in \mathbb{Z}$ such that

$$\left| \begin{pmatrix} x - a & b \\ b & x - c \end{pmatrix} \right| = x^2 - 3$$

so there cannot be an ISM A such that $R_A(z) = z^4 - z^2 + 1$.

"Missing" Mahler Measures

It would suffice to show that for any monic reciprocal $P \in \mathbb{Z}[z]$, there is an ISM A with

$$M(P) = M(A) \geq \lambda_0$$

"Missing" Mahler Measures

It would suffice to show that for any monic reciprocal $P \in \mathbb{Z}[z]$, there is an ISM A with

$$M(P) = M(A) \geq \lambda_0$$

Corollary (McKee, Smyth)

If A is an ISM with $1 < M(A) < 1.22$, then

$$M(A) \in \{1.17628\ldots, 1.18837\ldots, 1.20003\ldots, 1.21639\ldots, 1.21972\ldots\}$$

"Missing" Mahler Measures

It would suffice to show that for any monic reciprocal $P \in \mathbb{Z}[z]$, there is an ISM A with

$$M(P) = M(A) \geq \lambda_0$$

Corollary (McKee, Smyth)

If A is an ISM with $1 < M(A) < 1.22$, then

$$M(A) \in \{1.17628\ldots, 1.18837\ldots, 1.20003\ldots, 1.21639\ldots, 1.21972\ldots\}$$

But $P = z^{14} - z^{12} + z^7 - z^2 + 1$ has $M(P) = 1.20261\ldots$.

Cyclotomic Matrices over $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree

For A a Hermitian R-matrix:

Cyclotomic Matrices over $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree

For A a Hermitian R-matrix:

- $R_A(z) \in \mathbb{Z}[z]$.

Cyclotomic Matrices over $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree

For A a Hermitian R-matrix:

- $R_A(z) \in \mathbb{Z}[z]$.
- Interlacing Theorem still applies.

Cyclotomic Matrices over $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree

For A a Hermitian R-matrix:

- $R_A(z) \in \mathbb{Z}[z]$.
- Interlacing Theorem still applies.
- So A cyclotomic $\Rightarrow A_{i,i} \in \{0, \pm 1, \pm 2\}$.

Cyclotomic Matrices over $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree

For A a Hermitian R-matrix:

- $R_A(z) \in \mathbb{Z}[z]$.
- Interlacing Theorem still applies.
- So A cyclotomic $\Rightarrow A_{i,i} \in \{0, \pm 1, \pm 2\}$.
- A cyclotomic $\Rightarrow A_{i,j}A_{j,i} = Norm(A_{i,j}) \leq 4$.

Weighted Entries

Let $\mathcal{L}_n = \{x \in R \,|\, Norm(x) = n\}$ and define

$$\mathcal{L} := \mathcal{L}_4 \cup \mathcal{L}_3 \cup \mathcal{L}_2 \cup \mathcal{L}_1 \cup \{0\}.$$

Weighted Entries

Let $\mathcal{L}_n = \{x \in R \,|\, Norm(x) = n\}$ and define

$$\mathcal{L} := \mathcal{L}_4 \cup \mathcal{L}_3 \cup \mathcal{L}_2 \cup \mathcal{L}_1 \cup \{0\}.$$

- A a cyclotomic R-matrix $\Rightarrow$ A is an $\mathcal{L}$-matrix.

Weighted Entries

Let $\mathcal{L}_n = \{x \in R \,|\, Norm(x) = n\}$ and define

$$\mathcal{L} := \mathcal{L}_4 \cup \mathcal{L}_3 \cup \mathcal{L}_2 \cup \mathcal{L}_1 \cup \{0\}.$$

- A a cyclotomic R-matrix $\Rightarrow$ A is an $\mathcal{L}$-matrix.
- If $d < -15$, squarefree then $\mathcal{L} = \{0, \pm 1, \pm 2\}$, so A a cyclotomic R-matrix $\Rightarrow$ A an ISM.

Weighted Entries

Let $\mathcal{L}_n = \{x \in R \,|\, Norm(x) = n\}$ and define

$$\mathcal{L} := \mathcal{L}_4 \cup \mathcal{L}_3 \cup \mathcal{L}_2 \cup \mathcal{L}_1 \cup \{0\}.$$

- A a cyclotomic R-matrix $\Rightarrow$ A is an $\mathcal{L}$-matrix.
- If $d < -15$, squarefree then $\mathcal{L} = \{0, \pm 1, \pm 2\}$, so A a cyclotomic R-matrix $\Rightarrow$ A an ISM.
- Can restrict our attention to $d \in \{-1, -2, -3, -7, -11, -15\}$.

Weighted Entries

Let $\mathcal{L}_n = \{x \in R \,|\, Norm(x) = n\}$ and define

$$\mathcal{L} := \mathcal{L}_4 \cup \mathcal{L}_3 \cup \mathcal{L}_2 \cup \mathcal{L}_1 \cup \{0\}.$$

- A a cyclotomic R-matrix $\Rightarrow$ A is an $\mathcal{L}$-matrix.
- If $d < -15$, squarefree then $\mathcal{L} = \{0, \pm 1, \pm 2\}$, so A a cyclotomic R-matrix $\Rightarrow$ A an ISM.
- Can restrict our attention to $d \in \{-1, -2, -3, -7, -11, -15\}$.
- Strong equivalence generalises to conjugation by $O_n(\mathcal{L}_1)$.

Weighted Entries

Let $\mathcal{L}_n = \{x \in R \,|\, Norm(x) = n\}$ and define

$$\mathcal{L} := \mathcal{L}_4 \cup \mathcal{L}_3 \cup \mathcal{L}_2 \cup \mathcal{L}_1 \cup \{0\}.$$

- A a cyclotomic R-matrix $\Rightarrow$ A is an $\mathcal{L}$-matrix.
- If $d < -15$, squarefree then $\mathcal{L} = \{0, \pm 1, \pm 2\}$, so A a cyclotomic R-matrix $\Rightarrow$ A an ISM.
- Can restrict our attention to $d \in \{-1, -2, -3, -7, -11, -15\}$.
- Strong equivalence generalises to conjugation by $O_n(\mathcal{L}_1)$.
- Equivalence generalises to strong equivalence to any of $A, -A, \overline{A}, -\overline{A}$.

Weight 4 entries

Lemma

If A is a maximal indecomposable cyclotomic $\mathcal{L}$-matrix and contains an entry of weight 4 (i.e., from $\mathcal{L}_4$), then A is equivalent to one of

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{3}{2}+\frac{\sqrt{-7}}{2} \\ \frac{3}{2}-\frac{\sqrt{-7}}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2}+\frac{\sqrt{-15}}{2} \\ \frac{1}{2}-\frac{\sqrt{-15}}{2} & 0 \end{pmatrix}$$

Weight 4 entries

Lemma

If A is a maximal indecomposable cyclotomic $\mathcal{L}$-matrix and contains an entry of weight 4 (i.e., from $\mathcal{L}_4$), then A is equivalent to one of

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{3}{2} + \frac{\sqrt{-7}}{2} \\ \frac{3}{2} - \frac{\sqrt{-7}}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} + \frac{\sqrt{-15}}{2} \\ \frac{1}{2} - \frac{\sqrt{-15}}{2} & 0 \end{pmatrix}$$

(This is the only new class for $d = -15$.)

Weighted Labels

Hermitian $\mathcal{L}$-matrix

Charged $\mathcal{L}$-graph

Weighted Labels

Hermitian $\mathcal{L}$-matrix

- $A_{ii} \in \{0, 1, -1\}$.

Charged $\mathcal{L}$-graph

Weighted Labels

Hermitian $\mathcal{L}$-matrix

- $A_{ii} \in \{0, 1, -1\}$.

Charged $\mathcal{L}$-graph

- Vertex i ●, ⊕ or ⊖.

Weighted Labels

Hermitian $\mathcal{L}$-matrix

- $A_{ii} \in \{0, 1, -1\}$.
- $A_{ij} \in \mathcal{L}_1$.

Charged $\mathcal{L}$-graph

- Vertex i ●, ⊕ or ⊖.

Weighted Labels

Hermitian $\mathcal{L}$-matrix

- $A_{ii} \in \{0, 1, -1\}$.
- $A_{ij} \in \mathcal{L}_1$.

Charged $\mathcal{L}$-graph

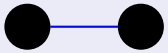
- Vertex i ●, ⊕ or ⊖.
- Weight 1 Edge: ●—●.

Weighted Labels

Hermitian $\mathcal{L}$-matrix

- $A_{ii} \in \{0, 1, -1\}$.
- $A_{ij} \in \mathcal{L}_1$.
- $A_{ij} \in \mathcal{L}_2$.

Charged $\mathcal{L}$-graph

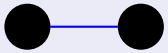
- Vertex i ●, ⊕ or ⊖.
- Weight 1 Edge: .

Weighted Labels

Hermitian $\mathcal{L}$-matrix

- $A_{ii} \in \{0, 1, -1\}$.
- $A_{ij} \in \mathcal{L}_1$.
- $A_{ij} \in \mathcal{L}_2$.

Charged $\mathcal{L}$-graph

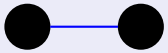
- Vertex i ●, ⊕ or ⊖.
- Weight 1 Edge: .
- Weight 2 Edge: .

Weighted Labels

Hermitian $\mathcal{L}$-matrix

- $A_{ii} \in \{0, 1, -1\}$.
- $A_{ij} \in \mathcal{L}_1$.
- $A_{ij} \in \mathcal{L}_2$.
- $A_{ij} \in \mathcal{L}_3$.

Charged $\mathcal{L}$-graph

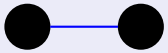
- Vertex i ●, ⊕ or ⊖.
- Weight 1 Edge: .
- Weight 2 Edge: .

Weighted Labels

Hermitian $\mathcal{L}$-matrix

- $A_{ii} \in \{0, 1, -1\}$.
- $A_{ij} \in \mathcal{L}_1$.
- $A_{ij} \in \mathcal{L}_2$.
- $A_{ij} \in \mathcal{L}_3$.

Charged $\mathcal{L}$-graph

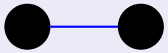
- Vertex i ●, ⊕ or ⊖.
- Weight 1 Edge: .
- Weight 2 Edge: .
- Weight 3 Edge: .

Weighted Labels

Hermitian $\mathcal{L}$-matrix

- $A_{ii} \in \{0, 1, -1\}$.
- $A_{ij} \in \mathcal{L}_1$.
- $A_{ij} \in \mathcal{L}_2$.
- $A_{ij} \in \mathcal{L}_3$.

Charged $\mathcal{L}$-graph

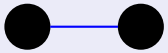
- Vertex i ●, $\oplus$ or $\ominus$.
- Weight 1 Edge: .
- Weight 2 Edge: .
- Weight 3 Edge: .
- Weighted degree of vertex i.

Weighted Labels

Hermitian $\mathcal{L}$-matrix

- $A_{ii} \in \{0, 1, -1\}$.
- $A_{ij} \in \mathcal{L}_1$.
- $A_{ij} \in \mathcal{L}_2$.
- $A_{ij} \in \mathcal{L}_3$.
- $\sum_{k=1}^{n} Norm(A_{i,k})$.

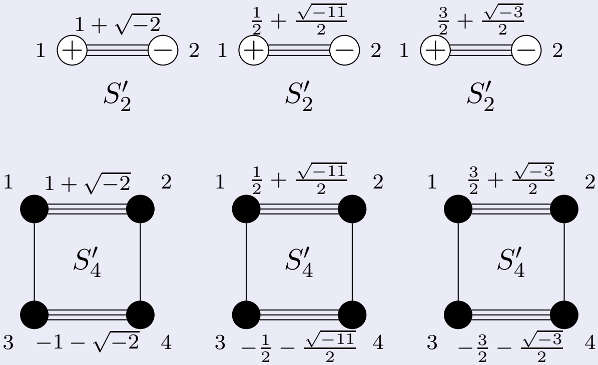
Charged $\mathcal{L}$-graph

- Vertex i ●, ⊕ or ⊖.
- Weight 1 Edge: .
- Weight 2 Edge: .
- Weight 3 Edge: .
- Weighted degree of vertex i.

$\mathcal{L}$-graphs with weight 3 edges

Lemma

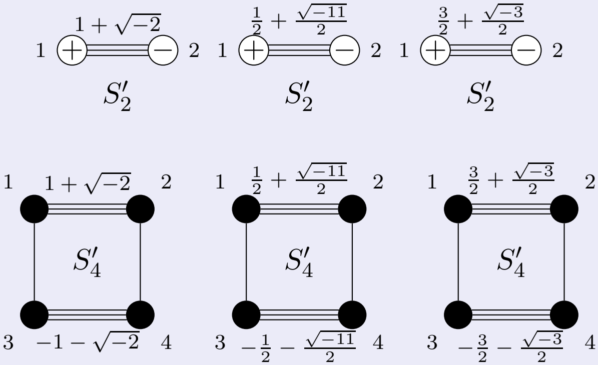
If G is a maximal connected cyclotomic $\mathcal{L}$-graph with an edge label from $\mathcal{L}_3$ then G is equivalent to one of



$\mathcal{L}$-graphs with weight 3 edges

Lemma

If G is a maximal connected cyclotomic $\mathcal{L}$-graph with an edge label from $\mathcal{L}_3$ then G is equivalent to one of



(These are the only new classes for $d=-11$.)

4-cyclotomic $\mathcal{L}$-graphs

Definition

A cyclotomic $\mathcal{L}$-graph is 4-cyclotomic if each vertex has weighted degree 4.

4-cyclotomic $\mathcal{L}$-graphs

Definition

A cyclotomic $\mathcal{L}$-graph is 4-cyclotomic if each vertex has weighted degree 4.

Lemma

'4-cyclotomic' $\Rightarrow$ 'Maximal Cyclotomic'.

Maximal cyclotomic $= 4$-cyclotomic?

By inspection, ‘maximal cyclotomic’$\Leftrightarrow$ ‘4-cyclotomic’ for:

Maximal cyclotomic $=4$-cyclotomic?

By inspection, 'maximal cyclotomic'$\Leftrightarrow$ '4-cyclotomic' for:

- Charged signed graphs.

Maximal cyclotomic = 4-cyclotomic?

By inspection, 'maximal cyclotomic'$\Leftrightarrow$ '4-cyclotomic' for:

- Charged signed graphs.
- $\mathcal{L}$-graphs, $d = -15$.

Maximal cyclotomic $=$ 4-cyclotomic?

By inspection, 'maximal cyclotomic'$\Leftrightarrow$ '4-cyclotomic' for:

- Charged signed graphs.
- $\mathcal{L}$-graphs, $d = -15$.
- $\mathcal{L}$-graphs, $d = -11$.

Maximal cyclotomic = 4-cyclotomic?

By inspection, ‘maximal cyclotomic’$\Leftrightarrow$ ‘4-cyclotomic’ for:

- Charged signed graphs.
- $\mathcal{L}$-graphs, $d = -15$.
- $\mathcal{L}$-graphs, $d = -11$.

Theorem

For $d = -2, -7$, an $\mathcal{L}$-graph is maximal cyclotomic if and only if it is 4-cyclotomic.

Maximal cyclotomic = 4-cyclotomic?

By inspection, 'maximal cyclotomic'$\Leftrightarrow$ '4-cyclotomic' for:

- Charged signed graphs.
- $\mathcal{L}$-graphs, $d = -15$.
- $\mathcal{L}$-graphs, $d = -11$.

Theorem

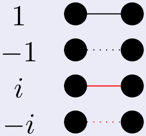
For $d = -2, -7$, an $\mathcal{L}$-graph is maximal cyclotomic if and only if it is 4-cyclotomic.

Theorem (Greaves)

For $d = -1, -3$, an $\mathcal{L}$-graph is maximal cyclotomic if and only if it is 4-cyclotomic.

More notation

$\mathcal{L}_1$, $d = -1$



More notation

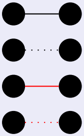
$\mathcal{L}_1$, $d = -1$

1

-1

i

$-i$



$\mathcal{L}_1$, $d = -3$

1

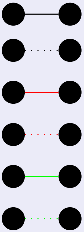
-1

$\frac{1}{2} + \frac{\sqrt{-3}}{2}$

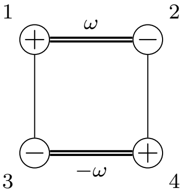
$-\frac{1}{2} - \frac{\sqrt{-3}}{2}$

$\frac{1}{2} - \frac{\sqrt{-3}}{2}$

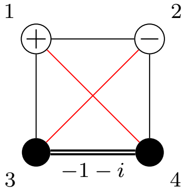
$-\frac{1}{2} + \frac{\sqrt{-3}}{2}$



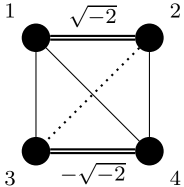
$\mathcal{L}$-graphs with weight 2 edges



S_4 $(d = -1, -2, -7)$

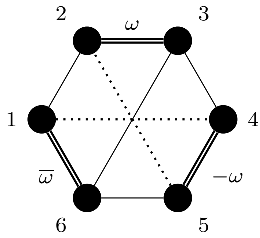


$S_4^\dagger$ $(d = -1)$

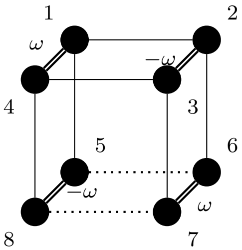


S_4^* $(d = -2)$

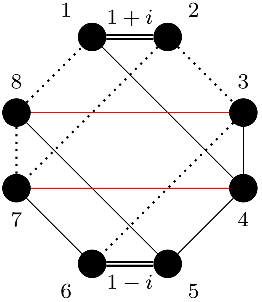
$\mathcal{L}$-graphs with weight 2 edges



$S_6^\dagger$ $(d = -7)$, $\omega = \frac{1}{2} + \frac{\sqrt{-7}}{2}$

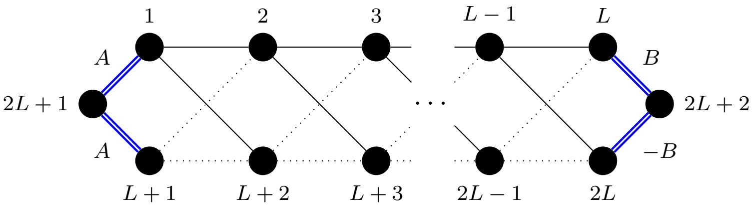


S_8^* $(d = -1, -2, -7)$



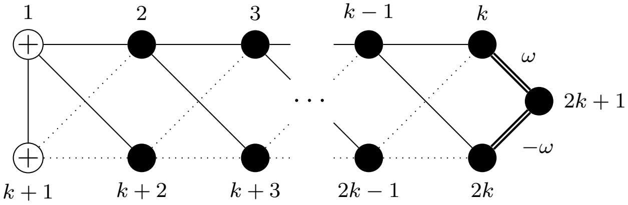
$S_8^\ddagger$ $(d = -1)$

$\mathcal{L}$-graphs with weight 2 edges



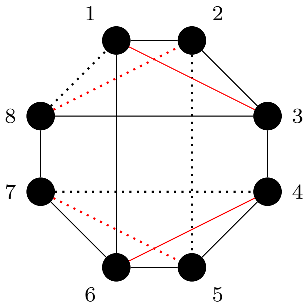
- T^4_{2k}, $k \geq 3$, $d = -1$, $A = B = 1 + i$.
- T^4_{2k}, $k \geq 3$, $d = -2$, $A = B = \sqrt{-2}$.
- T^4_{2k}, $k \geq 3$, $d = -7$, $A = B = \omega = \frac{1}{2} + \frac{\sqrt{-7}}{2}$.
- ${T^4_{2k}}'$, $k \geq 3$, $d = -7$, $A = \omega, B = \overline{\omega}$.

$\mathcal{L}$-graphs with weight 2 edges



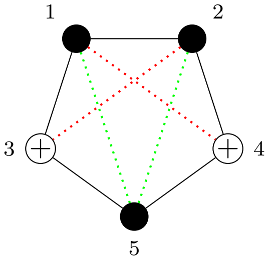
The $2k+1$-vertex $\mathcal{L}$-graphs C_{2k}^{2+}, $k \geq 1$, $d = -1, -2, -7$.

$\mathcal{L}$-graphs with all edges weight 1

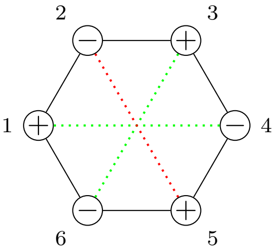


$S_8^{\dagger}$ $(d = -1)$

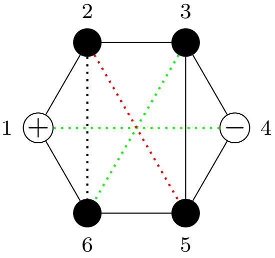
$\mathcal{L}$-graphs with all edges weight 1



S_5 $(d = -3)$

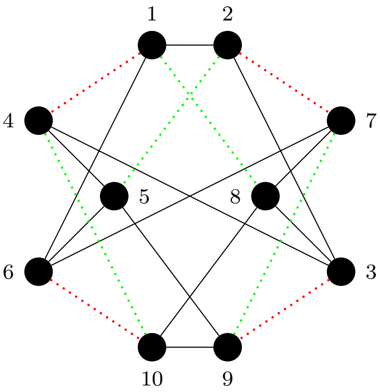


S_6 $(d = -3)$

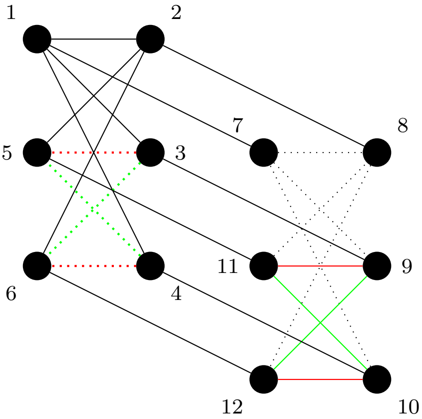


S_6' $(d = -3)$

$\mathcal{L}$-graphs with all edges weight 1

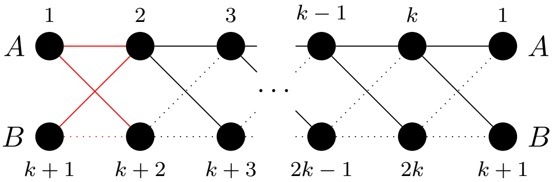


S_{10}, $(d = -3)$

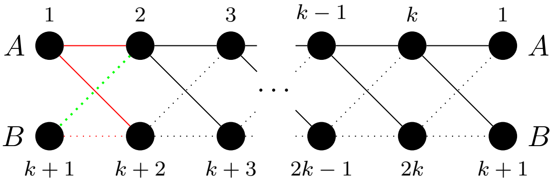


S_{12} $(d = -3)$

$\mathcal{L}$-graphs with all edges weight 1



T'_{2k}, $k \geq 3$ $(d = -1)$



T'_{2k}, $k \geq 3$ $(d = -3)$

Minimal Noncyclotomics

Definition

A noncyclotomic $\mathcal{L}$-graph is *minimal* if every proper induced subgraph is cyclotomic.

Minimal Noncyclotomics

Definition

A noncyclotomic $\mathcal{L}$-graph is *minimal* if every proper induced subgraph is cyclotomic.

- If G' is an induced subgraph of G, then $M(G) \geq M(G')$ (interlacing).

Minimal Noncyclotomics

Definition

A noncyclotomic $\mathcal{L}$-graph is *minimal* if every proper induced subgraph is cyclotomic.

- If G' is an induced subgraph of G, then $M(G) \geq M(G')$ (interlacing).
- Every noncyclotomic $\mathcal{L}$-graph has a minimal noncyclotomic $\mathcal{L}$-graph as a subgraph.

Minimal Noncyclotomics

Let $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree, $d \neq -1, -3$ and G a minimal noncyclotomic R-graph. Then G is an:

Minimal Noncyclotomics

Let $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree, $d \neq -1, -3$ and G a minimal noncyclotomic R-graph. Then G is an:

	# Vertices	$M(G)$
R-graph with a 'large' charge	1	> 2.618

Minimal Noncyclotomics

Let $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree, $d \neq -1, -3$ and G a minimal noncyclotomic R-graph. Then G is an:

	# Vertices	$M(G)$
R-graph with a 'large' charge	1	> 2.618
R-graph with a weight n edge, $n \geq 5$	2	> 2.36

Minimal Noncyclotomics

Let $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree, $d \neq -1, -3$ and G a minimal noncyclotomic R-graph. Then G is an:

	# Vertices	$M(G)$
R-graph with a 'large' charge	1	> 2.618
R-graph with a weight n edge, $n \geq 5$	2	> 2.36
R-graph with a ± 2 charge	2	> 1.722

Minimal Noncyclotomics

Let $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree, $d \neq -1, -3$ and G a minimal noncyclotomic R-graph. Then G is an:

	# Vertices	$M(G)$
R-graph with a 'large' charge	1	> 2.618
R-graph with a weight n edge, $n \geq 5$	2	> 2.36
R-graph with a ± 2 charge	2	> 1.722
$\mathcal{L}$-graph with a weight 4 edge	≤ 3	> 2.08

Minimal Noncyclotomics

Let $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree, $d \neq -1, -3$ and G a minimal noncyclotomic R-graph. Then G is an:

	# Vertices	$M(G)$
R-graph with a 'large' charge	1	> 2.618
R-graph with a weight n edge, $n \geq 5$	2	> 2.36
R-graph with a ± 2 charge	2	> 1.722
$\mathcal{L}$-graph with a weight 4 edge	≤ 3	> 2.08
$\mathcal{L}$-graph with a weight 3 edge	≤ 4	> 1.56

Minimal Noncyclotomics

Let $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree, $d \neq -1, -3$ and G a minimal noncyclotomic R-graph. Then G is an:

	# Vertices	$M(G)$
R-graph with a 'large' charge	1	> 2.618
R-graph with a weight n edge, $n \geq 5$	2	> 2.36
R-graph with a ± 2 charge	2	> 1.722
$\mathcal{L}$-graph with a weight 4 edge	≤ 3	> 2.08
$\mathcal{L}$-graph with a weight 3 edge	≤ 4	> 1.56
$\mathcal{L}$-graph with a weight 2 edge	≤ 10	> 1.35

Minimal Noncyclotomics

Let $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree, $d \neq -1, -3$ and G a minimal noncyclotomic R-graph. Then G is an:

	# Vertices	$M(G)$
R-graph with a 'large' charge	1	> 2.618
R-graph with a weight n edge, $n \geq 5$	2	> 2.36
R-graph with a ± 2 charge	2	> 1.722
$\mathcal{L}$-graph with a weight 4 edge	≤ 3	> 2.08
$\mathcal{L}$-graph with a weight 3 edge	≤ 4	> 1.56
$\mathcal{L}$-graph with a weight 2 edge	≤ 10	> 1.35
or a charged signed graph	≤ 10	$\geq \lambda_0$

Lehmer's Problem for (some) R-matrices

Theorem

If A is a R-matrix for $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree, $d \neq -1, -3$ then

$$M(A) \geq \lambda_0$$

Current Work

- Search for small minimal noncyclotomic $\mathcal{L}$-graphs:

Current Work

- Search for small minimal noncyclotomic $\mathcal{L}$-graphs:
  - $d = -1, -3$ $\mathcal{L}$-graphs with edges of weight $n \geq 3$ or large charges - finite set.

Current Work

- Search for small minimal noncyclotomic $\mathcal{L}$-graphs:
  - $d = -1, -3$ $\mathcal{L}$-graphs with edges of weight $n \geq 3$ or large charges - finite set.
  - $\{\pm 1, \pm\omega, \pm\overline{\omega}\}$-graphs - potentially infinite, tested to ten vertices.

Current Work

- Search for small minimal noncyclotomic $\mathcal{L}$-graphs:
  - $d = -1, -3$ $\mathcal{L}$-graphs with edges of weight $n \geq 3$ or large charges - finite set.
  - $\{\pm 1, \pm\omega, \pm\overline{\omega}\}$-graphs - potentially infinite, tested to ten vertices.
  - $\{\pm 1, \pm i\}$-graphs - potentially infinite, tested to ten vertices.

Current Work

- Search for small minimal noncyclotomic $\mathcal{L}$-graphs:
  - $d = -1, -3$ $\mathcal{L}$-graphs with edges of weight $n \geq 3$ or large charges - finite set.
  - $\{\pm 1, \pm \omega, \pm \overline{\omega}\}$-graphs - potentially infinite, tested to ten vertices.
  - $\{\pm 1, \pm i\}$-graphs - potentially infinite, tested to ten vertices.
  - $\{\pm 1, \pm i, \pm 1 \pm i\}$-graphs - potentially infinite, tested to nine vertices.

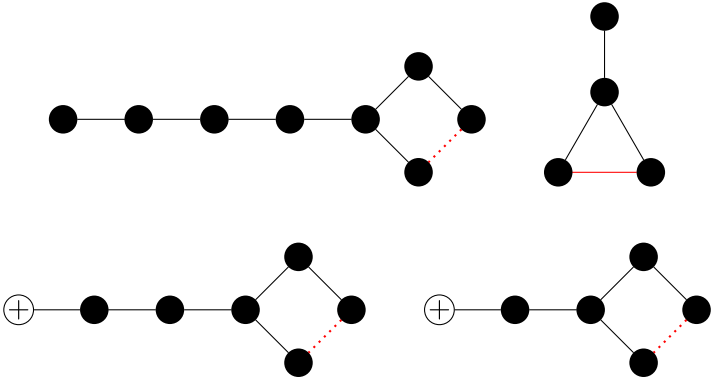
Current Work

- Search for small minimal noncyclotomic $\mathcal{L}$-graphs:
  - $d = -1, -3$ $\mathcal{L}$-graphs with edges of weight $n \geq 3$ or large charges - finite set.
  - $\{\pm 1, \pm\omega, \pm\overline{\omega}\}$-graphs - potentially infinite, tested to ten vertices.
  - $\{\pm 1, \pm i\}$-graphs - potentially infinite, tested to ten vertices.
  - $\{\pm 1, \pm i, \pm 1 \pm i\}$-graphs - potentially infinite, tested to nine vertices.
- Conjecture:

$$G \text{ an } n\text{-vertex minimal noncyclotomic } \mathcal{L}\text{-graph} \Rightarrow n \leq 10.$$

Sneak Preview

Minimal Noncyclotomic $\{\pm 1, \pm\omega, \pm\overline{\omega}\}$-graphs with small Mahler measure (that aren't just charged signed graphs in disguise):



Sneak Preview

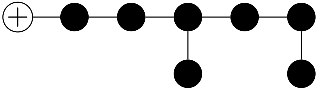
Let $P(z) = z^{14} - z^{13} - z^8 + z^7 - z^6 - z + 1$, with $M(P) = 1.267\ldots$.

Sneak Preview

Let $P(z) = z^{14} - z^{13} - z^8 + z^7 - z^6 - z + 1$, with $M(P) = 1.267\ldots$.
The only CSGs with Mahler Measure $1.267\ldots$ are nonminimal:

Sneak Preview

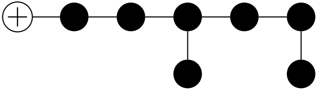
Let $P(z) = z^{14} - z^{13} - z^8 + z^7 - z^6 - z + 1$, with $M(P) = 1.267\ldots$.
The only CSGs with Mahler Measure $1.267\ldots$ are nonminimal:



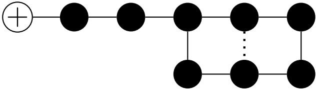
$$R_A = (z^2 + 1)P;$$

Sneak Preview

Let $P(z) = z^{14} - z^{13} - z^8 + z^7 - z^6 - z + 1$, with $M(P) = 1.267\ldots$.
The only CSGs with Mahler Measure $1.267\ldots$ are nonminimal:



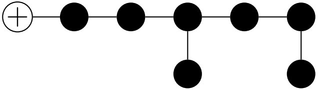
$R_A = (z^2 + 1)P$;



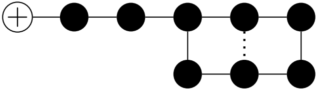
$R_A = (z^4 - z^2 + 1)P$;

Sneak Preview

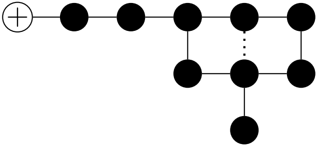
Let $P(z) = z^{14} - z^{13} - z^8 + z^7 - z^6 - z + 1$, with $M(P) = 1.267\ldots$.
The only CSGs with Mahler Measure $1.267\ldots$ are nonminimal:



$$R_A = (z^2 + 1)P;$$



$$R_A = (z^4 - z^2 + 1)P;$$



$$R_A = (z - 1)^2(z + 1)^2(z^2 + 1)P$$

Sneak Preview

So,

$$M(P) \in \{M(R_A(z)) \,|\, A \text{ a } \mathbb{Z}\text{-matrix}\}$$

but

$$P \notin \{R_A(z) \,|\, A \text{ a } \mathbb{Z}\text{-matrix}\}$$

Sneak Preview

So,

$$M(P) \in \{M(R_A(z)) \mid A \text{ a } \mathbb{Z}\text{-matrix}\}$$

but

$$P \notin \{R_A(z) \mid A \text{ a } \mathbb{Z}\text{-matrix}\}$$

However,

$$P \in \{R_A(z) \mid A \text{ a } \mathbb{Z}[\omega]\text{-matrix}\} :$$

Sneak Preview

So,

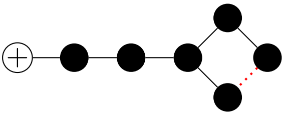
$$M(P) \in \{M(R_A(z)) \mid A \text{ a } \mathbb{Z}\text{-matrix}\}$$

but

$$P \notin \{R_A(z) \mid A \text{ a } \mathbb{Z}\text{-matrix}\}$$

However,

$$P \in \{R_A(z) \mid A \text{ a } \mathbb{Z}[\omega]\text{-matrix}\} :$$



$$R_A = P;$$

Further Reading

http://maths.straylight.co.uk