Cyclotomic Matrices and Graphs

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HIMR

8th September 2011

Mahler Measure

Let $P(z) = z^d + \cdots + a_d = \prod_{i=1}^{d}(z - \alpha_i) \in \mathbb{Z}[z]$ be monic, non-constant.

Definition

The *Mahler Measure* $M(P)$ is given by

$$M(P) := \exp\left(\int_0^1 \log|P(e^{2\pi it})|dt\right)$$

Mahler Measure

Equivalently,

$$M(P) = \prod_{i=1}^{d} \max\left(1, |\alpha_i|\right)$$

- Clearly, $M(P) \geq 1$ for all P.
- $M(P) = 1 \Leftrightarrow$ all nonzero roots of P are roots of unity.

Lehmer's Conjecture

In 1933, DH Lehmer exhibited the polynomial

$$z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

which has Mahler measure

$$\lambda_0 = 1.176280818\ldots$$

Lehmer's Conjecture

No-one has found a polynomial with smaller Mahler measure since!

- Lehmer's Problem: For a polynomial with $M(P) > 1$, can $M(P)$ be arbitrarily close to 1?
- If not, then there exists some $\lambda > 1$ such that $M(P) > 1 \Rightarrow M(P) \geq \lambda$.

Lehmer's Conjecture, strong version

$$M(P) > 1 \Rightarrow M(P) \geq \lambda_0$$

Associated Polynomials

- If A is an $n \times n$ integer symmetric matrix, then its *associated polynomial* is $R_A(z) := z^n \chi_A(z + 1/z)$
- If A has all eigenvalues in $[-2, 2]$, then $M(R_A) = 1$
  (We describe A as a *cyclotomic matrix*.)

From cyclotomic to noncyclotomic

Theorem (Cauchy Interlacing Theorem)

Let A be a Hermitian $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Let B be obtained from A by deleting row i and column i from A. Then the eigenvalues $\mu_1 \leq \cdots \leq \mu_{n-1}$ of B interlace with those of A: that is,

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n$$

From cyclotomic to noncyclotomic

We can run this process in reverse. Let B be a cyclotomic matrix, so its eigenvalues satisfy

$$-2 \leq \mu_1 \leq \cdots \leq \mu_{n-1} \leq 2$$

Then if we 'grow' a matrix A from B by adding an extra row and column, we have by interlacing

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n$$

So

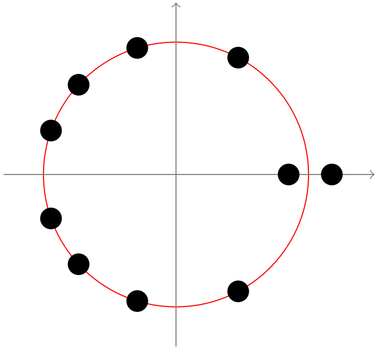
$$\lambda_2, \ldots, \lambda_{n-1} \in [\mu_1, \mu_{n-1}] \subseteq [-2, 2]$$

At worst,

$$\lambda_1, \lambda_n \notin [-2, 2]$$

Lehmer's Polynomial

λ_0 is a Salem number (all other conjugates satsify $|\alpha_i| \leq 1$, with at least one having absolute value 1):



Entries

Proposition

The only cyclotomic 1×1 *matrices are*

$$(0), (1), (-1), (2), (-2)$$

Corollary

By interlacing, the entries of an integer cyclotomic matrix must be elements of $\{0, 1, -1, 2, -2\}$.

Indecomposability

If M decomposes as a block-diagonal matrix, then its eigenvalues are those of the blocks. Thus a cyclotomic matrix decomposes into one or more indecomp. cyclotomic matrices.

Lemma

Apart from the matrices

$$(2), (-2), \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

any indecomp. cyclotomic matrix has all entries from $\{0, 1, -1\}$.

Maximality

- If A is cyclotomic, so is any B obtained by deleting some set of rows and corresponding columns of A: B is described as being *contained in* A.
- If A is an indecomp. cyclotomic matrix that is not contained in any strictly larger indecomp. cyclotomic matrix, then A is described as being *maximal*.

Theorem (McKee, Smyth)

Any non-maximal indecomp. cyclotomic matrix is contained in a maximal one.

Equivalence

Let $O_n(\mathbb{Z})$ be the orthogonal group of $n \times n$ signed permutation matrices, generated by permutation matrices and matrices of the form

$$diag(1, 1, \ldots, 1, -1, 1, \ldots, 1)$$

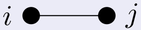
- If A is cyclotomic and $X \in O_n(\mathbb{Z})$, then $A' = XAX^{-1}$ is cyclotomic since it has the same eigenvalues. We describe A and A' as *strongly equivalent*.
- A matrix A' is then described as *equivalent* to A if it is strongly equivalent to either A or $-A$.

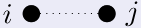
Charged Signed Graphs

Symmetric $\{0, 1, -1\}$-matrix

- $n \times n$ Matrix A
- $A_{ij} = 1$
- $A_{ij} = -1$
- $A_{ii} = 0$
- $A_{ii} = 1$
- $A_{ii} = -1$

Charged Signed Graph

- n-vertex Graph G
- Edge 

- Negative edge 

- Neutral Vertex ●
- Positive Vertex ⊕
- Negative Vertex ⊖

Charged Signed Graphs

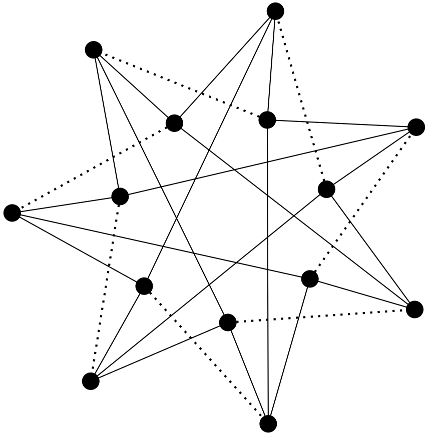
Symmetric $\{0, 1, -1\}$-matrix

- A indecomposable.
- A' contained in A.
- A_1 permutation of A_2.
- Conjugation by kth diagonal matrix.

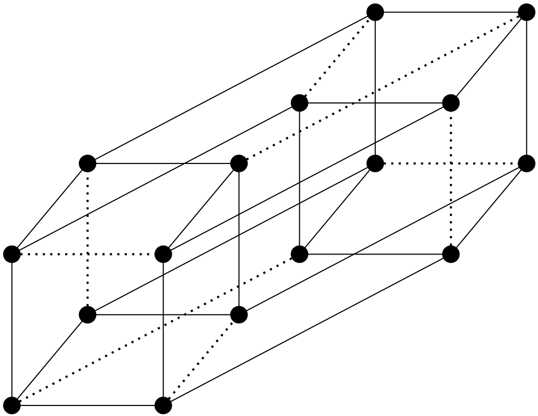
Charged Signed Graph

- G connected.
- G' induced subgraph of G.
- G_1 vertex relabelling of G_2.
- Switching of signs of all edges at vertex k.

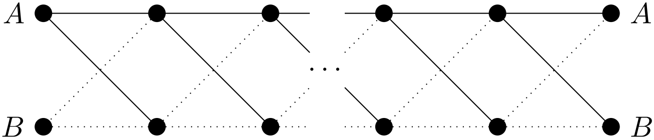
The maximal connected cyclotomic signed graph S_{14}.



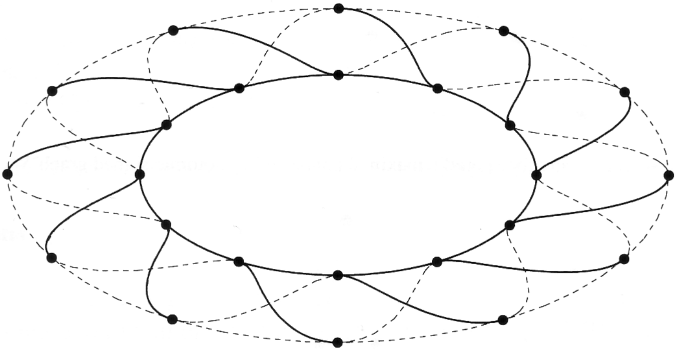
The maximal connected cyclotomic signed graph S_{16}.



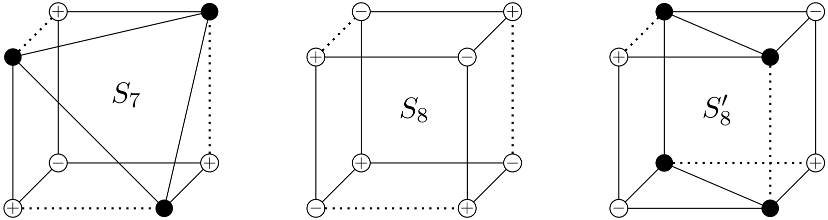
The maximal connected cyclotomic signed graphs T_{2k}.



Example: T_{24}

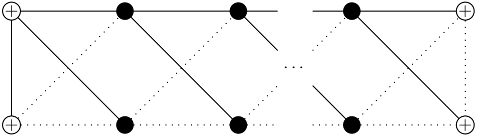


The maximal connected cyclotomic charged signed graphs S_7, S_8, S_8'.

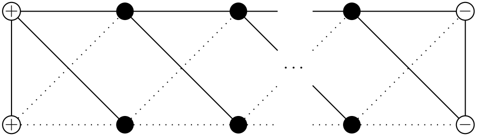


The maximal connected cyclotomic charged signed graphs C_{2k}^{++} and C_{2k}^{+-}.

$C_{2k}^{++} :=$



$C_{2k}^{+-} :=$



Minimal Noncyclotomics

Definition

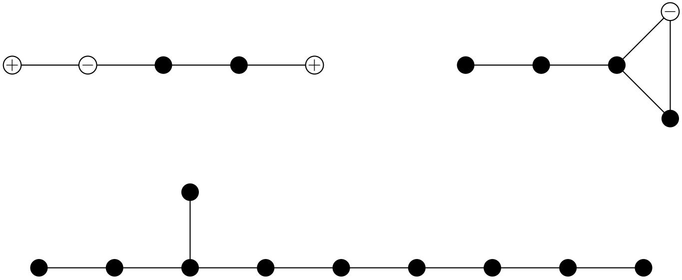
A noncyclotomic charged signed graph is *minimal* if every proper induced subgraph is cyclotomic.

- Every noncyclotomic CSG has a minimal noncyclotomic CSG as a subgraph.
- If G' is an induced subgraph of G, then $M(G) \geq M(G')$ (interlacing).

Minimal Noncyclotomics

Theorem (McKee,Smyth)

If G is a minimal noncyclotomic CSG, then G has at most ten vertices and Mahler measure at least λ_0.



Lehmer's Problem for ISMs

Theorem (McKee,Smyth)

If A is a noncyclotomic integer symmetric matrix then

$$M(R_A(z)) \geq \lambda_0$$

Does this solve Lehmer's Problem?

- Given $f \in \mathbb{Z}[z]$, does there exist an ISM A such that $R_A(z) = f$?
- Clearly, $R_A(z)$ is reciprocal. Fortunately:

Theorem (Smyth)

Let α be a nonreciprocal algebraic integer with minimal polynomial P_α. Then

$$M(\alpha) := M(P_\alpha) \geq M(z^3 - z - 1) = \theta_0 = 1.3247\ldots$$

- So we can safely restrict our attention to reciprocal f.

Missing Polynomials

- In fact, it'd suffice to find A such that $M(f) = M(A) \geq \lambda_0$.
- Sadly, even this is impossible:
  - $P = z^{14} - z^{12} + z^7 - z^2 + 1$ has $M(P) = 1.20261\ldots$.
  - But $1.20261\ldots \notin \{M(G) \,|\, G$ a CSG with $M(G) \leq 1.3\}$
  - So, $\nexists$ an ISM A such that $M(A) = M(P)$.

Imaginary Quadratic Extensions

Let $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ for $d < 0$ squarefree. Then For A a Hermitian R-matrix:

- $R_A(z) \in \mathbb{Z}[z]$.
- Interlacing Theorem still applies.
- A cyclotomic $\Rightarrow A_{i,j}A_{j,i} = Norm(A_{i,j}) \leq 4$.

$\mathcal{L}$-matrices

Let $\mathcal{L}_n = \{x \in R \,|\, Norm(x) = n\}$ and define

$$\mathcal{L} := \mathcal{L}_4 \cup \mathcal{L}_3 \cup \mathcal{L}_2 \cup \mathcal{L}_1 \cup \{0\}.$$

- A a cyclotomic R-matrix $\Rightarrow$ A is an $\mathcal{L}$-matrix.
- If $d < -15$, squarefree then $\mathcal{L} = \{0, \pm 1, \pm 2\}$, so

$$A \text{ a cyclotomic } R\text{-matrix} \Rightarrow A \text{ an ISM.}$$

- Can restrict our attention to $d \in \{-1, -2, -3, -7, -11, -15\}$.

$\mathcal{L}$-matrices

Proposition

If A is a maximal cyclotomic matrix with an entry of norm 4 or 3, then A is at most a 4×4 matrix.

Corollary

For $d = -15$ or -11, all but finitely many cyclotomic $\mathcal{L}$-matrices are ISMs (and the exceptions are easily determined).

$\mathcal{L}'$-matrices and graphs

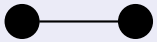
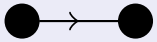
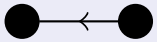
So, we can restrict our attention to $d \in \{-1, -2, -3, -7\}$, and $\mathcal{L}'$-matrices for $\mathcal{L}' = \mathcal{L}_2 \cup \mathcal{L}_1 \cup \{0\}$.

- Strong equivalence generalises to conjugation by elements of $U_n(R)$.
- Equivalence generalises to strong equivalence to any of $A, -A, \overline{A}, -\overline{A}$.

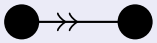
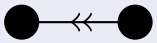
We can represent each class of matrices by an $\mathcal{L}'$-graph with edges of *weight* 1 or 2.

$\mathcal{L}'$-graphs

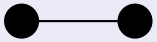
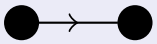
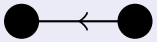
$\mathcal{L}_1$, $d = -1$

1	
-1	
i	
$-i$	

$\mathcal{L}_2$, $d = -1$

$1+i$	
$1-i$	
$-1-i$	
$-1+i$	

$\mathcal{L}_1$, $d = -3$

1	
-1	
$\omega = 1/2 + \sqrt{-3}/2$	
$-\omega$	
$\overline{\omega}$	
$-\overline{\omega}$	

$\mathcal{L}'$-graphs

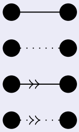
$\mathcal{L}'$, $d = -2$

1

-1

$\sqrt{-2}$

$-\sqrt{-2}$



$\mathcal{L}'$, $d = -7$

1

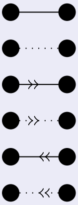
-1

$\omega = 1/2 + \sqrt{-7}/2$

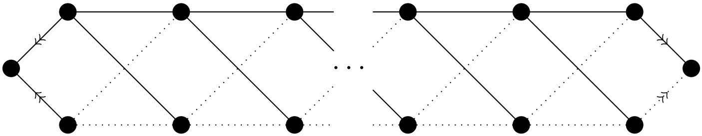
$-\omega$

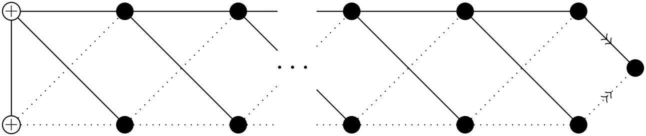
$\overline{\omega}$

$-\overline{\omega}$

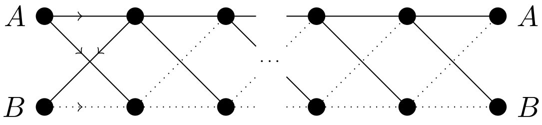


$\mathcal{L}'$-graphs with weight 2 edges

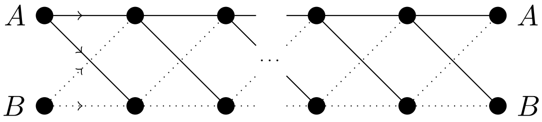




$\mathcal{L}$'-graphs with all edges weight 1



T'_{2k}, $k \geq 3$ $(d = -1)$



T'_{2k}, $k \geq 3$ $(d = -3)$

Extremal Cyclotomics

From the classification of ISMs, we have

Corollary

If M is a maximal cyclotomic ISM, then $M^2 = 4Id$.

We describe an R-matrix M satisfying $M^2 = 4Id$ as *extremal*.

Proposition

$$\textit{Extremal} \Rightarrow \textit{maximal cyclotomic}$$

Extremal Cyclotomics

By observation,

$$\text{Extremal} \Leftrightarrow \text{maximal cyclotomic}$$

for R any of

- $\mathbb{Z}$
- $\mathcal{O}_{\mathbb{Q}(\sqrt{-15})}$
- $\mathcal{O}_{\mathbb{Q}(\sqrt{-11})}$

Extremal Cyclotomics

Theorem

For $d = -2, -7$, if G is a cyclotomic $\mathcal{L}'$-graph with a vertex of weighted degree less than four, then there exists a cyclotomic $\mathcal{L}'$-graph H inducing G as a proper subgraph.

Classifying the maximal cyclotomic graphs (and hence cyclotomic matrices) for these rings therefore reduces to classifying the extremal graphs.

Classification of $\mathcal{L}'$-graphs

- Classified extremal $\mathcal{L}'$-graphs for $d \in \{-1, -2, -3, -7\}$. With finitely many (known) exceptions, any such graph is a CSG or in one of the infinite families given earlier.
- This completed the classification of cyclotomics for $d = -2, -7$ and gave a conjectural classification for $d = -1, -3$.
- Greaves (2011) classified the maximal cyclotomics for $d = -1, -3$, confirming the conjecture.

$d \neq -1, -3$

Theorem (2010)

For $d < 0$, squarefree, $d \neq -1, -3$: If G is a minimal noncyclotomic R-graph with $M(G) < 1.3$, then G is equivalent to a charged signed graph.

$d = -1, -3$

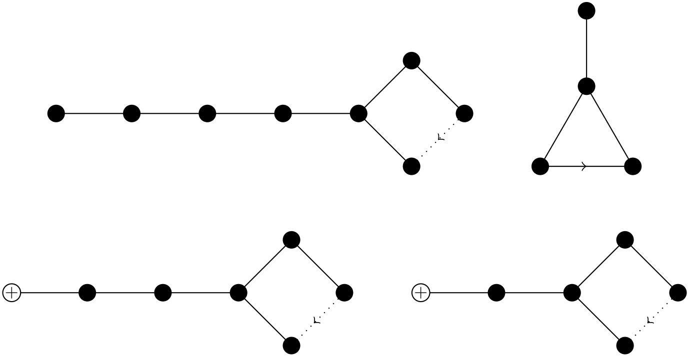
Theorem (2011, with Greaves)

If G is a minimal noncyclotomic $\mathbb{Z}[i]$-graph with $M(G) < 1.3$, then G is equivalent to a charged signed graph.

Theorem (2011, with Greaves)

If G is a minimal noncyclotomic $\mathbb{Z}[\omega]$-graph with $M(A) < 1.3$, then G is equivalent to a charged signed graph or one of four $\mathbb{Z}[\omega]$-graphs, which have Mahler measure at least $1.267\ldots$.

$\mathbb{Z}[\omega]$-graphs with $M(G) < 1.3$



Lehmer's Problem

Corollary (Good)

Let A be an R-matrix, for $R = \mathcal{O}_{\mathbb{Q}\sqrt{d}}$, $d < 0$ squarefree. Then

$$M(A) = 1 \text{ or } M(A) \geq \lambda_0$$

Corollary (Bad)

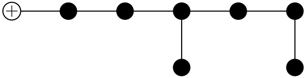
Recall $P = z^{14} - z^{12} + z^7 - z^2 + 1$ *with* $M(P) = 1.20261\ldots$.

$$1.20261\ldots \notin \{M(G) \,|\, G \text{ a } \mathbb{Z}[\omega]\text{-graph with } M(G) \leq 1.3\}.$$

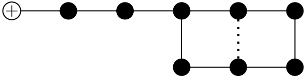
So there are still "missing" Mahler measures!

Consolation Prize

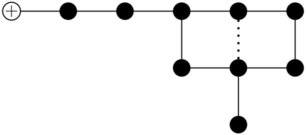
Let $P(z) = z^{14} - z^{13} - z^8 + z^7 - z^6 - z + 1$, with $M(P) = 1.267\ldots$.
The only CSGs with Mahler Measure $1.267\ldots$ are nonminimal:



$$R_A = (z^2 + 1)P;$$



$$R_A = (z^4 - z^2 + 1)P;$$



$$R_A = (z-1)^2(z+1)^2(z^2+1)P$$

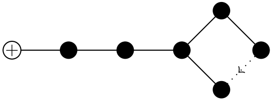
Consolation Prize

So

$$P \notin \{R_A(z) \,|\, A \text{ a } \mathbb{Z}\text{-matrix}\}$$

However,

$$P \in \{R_A(z) \,|\, A \text{ a } \mathbb{Z}[\omega]\text{-matrix}\} :$$



$$R_A = P;$$

(But $M(P) \in \{M(R_A(z)) \,|\, A \text{ a } \mathbb{Z}\text{-matrix}\}$)

Summary

- Classified all cyclotomic R-matrices (infinitely many new examples);
- Classified all minimal noncyclotomic R-matrices (new small Mahler measure graphs and associated polynomials);
- Lehmer's conjecture holds for associated polynomials of R-matrices;
- But, since that's not all of $\mathbb{Z}[z]$, remains open in general!

Further Reading

http://maths.straylight.co.uk